

Novel Adaptive Methods for Hyperbolic Conservation Laws Based on New Quasi-Linear Seventh- and Ninth-Order Schemes

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Abstract

We develop new adaptive numerical schemes for one- and two-dimensional hyperbolic systems of conservation laws. The methodology relies on the use of a smoothness indicator to automatically partition the computational domain into smooth and nonsmooth (“rough”) regions. We then follow the scheme adaption strategy recently introduced in [S. CHU, P. FENG, V. A. KOLOTILOV, A. KURGANOV, AND V. V. OSTAPENKO, Commun. Comput. Phys., accepted], but instead of the quasi-linear (QL) fifth-order finite-difference scheme used there, we employ the new QL seventh- and ninth-order schemes in the smooth regions. A series of numerical experiments for the Euler equations of gas dynamics demonstrates that the new adaptive schemes contain a smaller amount of numerical dissipation and achieve higher resolution compared with their counterpart that uses the QL fifth-order scheme in the smooth areas.

Key words: Adaptive schemes; low-dissipation central-upwind numerical fluxes; smoothness indicator; overcompressive and dissipative limiters; quasi-linear finite-difference schemes; Euler equations of gas dynamics.

AMS subject classification: 65M08, 65M06, 76M12, 76M20, 76L05, 35L65.

1 Introduction

In this paper, we consider hyperbolic systems of conservation laws, which in the one- (1-D) and two-dimensional (2-D) cases, read as

$$U_t + F(U)_x = 0, \quad (1.1)$$

and

$$U_t + F(U)_x + G(U)_y = 0, \quad (1.2)$$

respectively. Here, x and y are spatial variables, t is the time, $U \in \mathbb{R}^d$ is a vector of unknown functions, and $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $G : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are fluxes.

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It is well-known that solutions of (1.1) and (1.2) may develop complex structures consisting of shocks, rarefactions, and contact discontinuities. This complexity makes it difficult to simultaneously achieve (i) sharp, non-oscillatory resolution of discontinuities and (ii) high-order accuracy in smooth regions. For a variety of existing numerical methods for (1.1) and (1.2), we refer the reader to the monographs and review papers [1, 7, 8, 13, 21, 24] and references therein. Most of those methods rely on computationally expensive nonlinear limiters, which are, in fact, not needed to be applied in smooth regions.

In this paper, we focus on highly efficient scheme adaption methods, which are based on automatic detection of smooth and nonsmooth (“rough”) parts of the computed solutions and on applying different numerical methods in the corresponding parts of the computational domain. In particular, we follow our recent paper [3], where we have applied a smoothness indicator (SI) introduced in [16] to dynamically partition the computational domain into three parts: smooth, nonsmooth, and contact areas within the nonsmooth region. For the Euler equations of gas dynamics, this can be done by applying the SI to the density and pressure variables separately as pressure remains continuous at the contact surfaces, where the density jumps, while both the density and pressure jump across shock waves. After identifying the locations of contact surfaces, we apply the second-order low-dissipation central-upwind (LDCU) schemes introduced in [6] with an overcompressive limiter from [14] in their vicinities only, while utilizing the dissipative Minmod2 limiter in the rest of the “rough” areas. Both the limiters are applied to the local characteristic variables using the local characteristic decomposition, which is often used in the context of high-order schemes [18], but can also be implemented to suppress spurious oscillations, which might be otherwise developed by second-order schemes; see, e.g., [4, 5]. Finally, in the smooth regions, we have used the quasi-linear (QL) fifth-order scheme from [11], which is highly accurate and computationally inexpensive.

The main goal of the present paper is to further increase the resolution of the smooth parts of the computed solution. To this end, we replace the QL fifth-order schemes, which were used in [3], with the recently developed QL seventh- and ninth-order schemes from [10], and incorporate them into the same adaptive strategy. The new QL higher-order schemes rely on a high-order artificial-viscosity term whose strength is controlled by parameters that can be selected to optimize their linear stability properties. To determine the optimal values of those parameters, we conduct a von Neumann stability analysis for the 1-D QL seventh- and ninth-order schemes. In the 2-D extension, we follow a dimension-by-dimension construction and retain the same stability-motivated viscosity parameters. The resulting adaptive schemes preserve the robustness and sharp resolution of the discontinuities, while providing a higher-fidelity representation of smooth structures. The improvements are illustrated on a series of 1-D and 2-D benchmarks for the Euler equations of gas dynamics, where the seventh- and ninth-order schemes are compared against the fifth-order one within the same adaptive framework.

The rest of the paper is organized as follows. In §2, we present the 1-D r^{th} -order QL schemes, where $r \geq 3$ is an odd number. For $r = 7$ and 9, we study linear stability of these schemes, verify their orders of accuracy, and introduce the 1-D adaptive algorithms. In §3, we extend the r^{th} -order QL schemes to the 2-D case, and for $r = 7$ and 9, we introduce the 2-D adaptive algorithms. In §4, we report numerical experiments demonstrating the performance of the proposed schemes.

2 One-Dimensional Schemes

In this section, we consider the 1-D hyperbolic system (1.1) and describe the adaption algorithm, which is similar to the one recently introduced in [3]. We assume that the computational domain is covered with the uniform cells $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ with $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$ centered at $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$.

2.1 Quasi-Linear r^{th} -Order Schemes

We first briefly review the QL r^{th} -order schemes (with $r = 2\ell - 1$, $\ell = 2, 3, \dots$) recently proposed in [10]. These schemes are, in fact, r^{th} order in space and third order in time and they can be written in the following three stages:

$$U_j^{\text{I}} = U_j^n - \Delta t^n \mathcal{R}_j^n, \quad U_j^{\text{II}} = \frac{3}{4} U_j^n + \frac{1}{4} [U_j^{\text{I}} - \Delta t^n \mathcal{R}_j^{\text{I}}], \quad U_j^{n+1} = \frac{1}{3} U_j^n + \frac{2}{3} [U_j^{\text{II}} - \Delta t^n \mathcal{R}_j^{\text{II}}].$$

Here, Δt^n is a time step, $U_j^n \approx U(x_j, t^n)$, t^n is the current time level, $t^{n+1} = t^n + \Delta t^n$, and $\mathcal{R}_j^n$, $\mathcal{R}_j^{\text{I}}$, and $\mathcal{R}_j^{\text{II}}$ are the following discretizations of $F(U)_x$:

$$\mathcal{R}_j^n = (F_x)_j^n, \quad \mathcal{R}_j^{\text{I}} = (F_x)_j^{\text{I}}, \quad \mathcal{R}_j^{\text{II}} = (F_x)_j^{\text{II}} + (-1)^\ell \frac{3(\Delta x)^{r+1}}{2\Delta t^n} B(D_x^{r+1} U)_j^n. \quad (2.1)$$

Here, $(F_x)_j^n$ is a $(r+1)^{\text{st}}$ -order central-difference approximation of $F(U(x_j, t^n))_x$, which is given by

$$(F_x)_j^n = \frac{1}{\Delta x} \sum_{i=1}^{\ell} a_i (F(U_{j+i}^n) - F(U_{j-i}^n)) \quad (2.2)$$

with the coefficients a_i obtained by solving the following linear algebraic system:

$$\sum_{i=1}^{\ell} a_i i = \frac{1}{2}, \quad \sum_{i=1}^{\ell} a_i i^{2s+1} = 0, \quad s = 1, \dots, \ell - 1. \quad (2.3)$$

The terms $(F_x)_j^{\text{I}}$ and $(F_x)_j^{\text{II}}$ in (2.1) are determined similarly with the superscript n in (2.2) replaced with I and II, respectively.

Finally, $(D_x^{r+1} U)_j^n$ is the following second-order central-difference approximation of $\frac{\partial^{r+1}}{\partial x^{r+1}} U(x_j, t^n)$:

$$(D_x^{r+1} U)_j^n = \frac{1}{(\Delta x)^{r+1}} \sum_{i=-\ell}^{\ell} (-1)^{i+\ell} \binom{2\ell}{i+\ell} U_{j+i}^n,$$

which is a part of the term that represents high-order artificial viscosity used to stabilize the QL schemes. The amount of this artificial viscosity can be controlled by the positive parameter B in (2.1), which can be determined based on the linear stability analysis, which we conduct for the seventh- and ninth-order schemes in §2.2 and §2.3, respectively.

It is easy to show that the above QL r^{th} -order schemes can be written in the following flux form:

$$\begin{aligned} U_j^{\text{I}} &= U_j^n - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}} \right), \\ U_j^{\text{II}} &= \frac{3}{4} U_j^n + \frac{1}{4} \left[U_j^{\text{I}} - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2}}^{\text{I}} - \mathcal{F}_{j-\frac{1}{2}}^{\text{I}} \right) \right], \\ U_j^{n+1} &= \frac{1}{3} U_j^n + \frac{2}{3} \left[U_j^{\text{II}} - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2}}^{\text{II}} - \mathcal{F}_{j-\frac{1}{2}}^{\text{II}} \right) \right], \end{aligned} \quad (2.4)$$

where $\lambda^n := \Delta t^n / \Delta x$ and the numerical fluxes are defined by:

$$\mathcal{F}_{j+\frac{1}{2}} = \mathcal{L}_{j+\frac{1}{2}}[U^n], \quad \mathcal{F}_{j+\frac{1}{2}}^{\text{I}} = \mathcal{L}_{j+\frac{1}{2}}[U^{\text{I}}], \quad \mathcal{F}_{j+\frac{1}{2}}^{\text{II}} = \mathcal{L}_{j+\frac{1}{2}}[U^{\text{II}}] - \omega_{j+\frac{1}{2}}^n, \quad (2.5)$$

where

$$\mathcal{L}_{j+\frac{1}{2}}[U] := \sum_{i=1-\ell}^{\ell} b_i F(U_{j+i}) \quad \text{and} \quad \omega_{j+\frac{1}{2}}^n := \frac{3B}{2\lambda^n} \sum_{i=1-\ell}^{\ell} (-1)^{i+\ell} \binom{2\ell-1}{i+\ell-1} F(U_{j+i}).$$

Here, the coefficients

$$b_i = b_{1-i} = \sum_{m=i}^{\ell} a_m, \quad i = 1, \dots, \ell, \quad (2.6)$$

with a_m defined by (2.3).

2.2 Von Neumann Stability Analysis of the Quasi-Linear Seventh-Order Schemes

We consider the QL seventh-order schemes, which reads as (2.4)–(2.5) with

$$\begin{aligned}\mathcal{L}_{j+\frac{1}{2}}[U] &= \frac{1}{840} \left[-3\mathbf{F}(U_{j+4}) + 29\mathbf{F}(U_{j+3}) - 139\mathbf{F}(U_{j+2}) + 533\mathbf{F}(U_{j+1}) \right. \\ &\quad \left. + 533\mathbf{F}(U_j) - 139\mathbf{F}(U_{j-1}) + 29\mathbf{F}(U_{j-2}) - 3\mathbf{F}(U_{j-3}) \right], \\ \omega_{j+\frac{1}{2}}^n &= \frac{3}{512\lambda^n} \left[-U_{j+4}^n + 7U_{j+3}^n - 21U_{j+2}^n + 35U_{j+1}^n - 35U_j^n + 21U_{j-1}^n - 7U_{j-2}^n + U_{j-3}^n \right],\end{aligned}\tag{2.7}$$

and apply them to the linear advection equation

$$\psi_t + \psi_x = 0,\tag{2.8}$$

for which the speeds of propagation do not change and hence $\lambda^n \equiv \lambda$ as the size of time steps is independent of n , namely, $\Delta t^n \equiv \Delta t$. We then substitute the Fourier mode

$$\psi_j^n = g^n e^{ij\alpha}, \quad i = \sqrt{-1},\tag{2.9}$$

into the scheme (2.4)–(2.5), (2.7) and obtain the amplification factor

$$g = g(\lambda, B, \alpha) = \frac{1}{3} + \frac{2}{3} (1 + \lambda\delta(\alpha)) \left(1 + \frac{1}{2}\lambda\delta(\alpha) + \frac{1}{4}(\lambda)^2\delta(\alpha)^2 \right) + B\Theta(\alpha),\tag{2.10}$$

where

$$\begin{aligned}\delta(\alpha) &= \frac{1}{280} (e^{4i\alpha} - e^{-4i\alpha}) - \frac{4}{105} (e^{3i\alpha} - e^{-3i\alpha}) + \frac{1}{5} (e^{2i\alpha} - e^{-2i\alpha}) - \frac{4}{5} (e^{i\alpha} - e^{-i\alpha}) \\ &= i \left(\frac{1}{140} \sin(4\alpha) - \frac{8}{105} \sin(3\alpha) + \frac{2}{5} \sin(2\alpha) - \frac{8}{5} \sin \alpha \right), \\ \Theta(\alpha) &= -(e^{4i\alpha} + e^{-4i\alpha}) + 8(e^{3i\alpha} + e^{-3i\alpha}) - 28(e^{2i\alpha} + e^{-2i\alpha}) + 56(e^{i\alpha} + e^{-i\alpha}) - 70 \\ &= -2\cos(4\alpha) + 16\cos(3\alpha) - 56\cos(2\alpha) + 112\cos(\alpha) - 70.\end{aligned}\tag{2.11}$$

The QL seventh-order scheme is then stable provided

$$|g(\lambda, B, \alpha)| \leq 1, \quad \forall \alpha \in [0, 2\pi].$$

First, we note that if $B = 0$ then $g(\lambda, B, \pi) = 1$, and the resulting scheme will produce substantial oscillations when it is applied to an initial value problem with discontinuous initial data. We therefore need to use $B > 0$. To select a particular value of B , we make the following considerations. In Figure 2.1, we plot the values of g for $\lambda = 0.5$ (this is an upper bound for the CFL number for the LDCU schemes, which will be used in our adaptive methods) for different values of

$$\alpha = \alpha_\ell = \frac{\pi\ell}{20}, \quad \ell = 0, \dots, 40,\tag{2.12}$$

and $B = 0, 0.002, 0.006$, and 0.008 . As one can see, when $B = 0.008$, the scheme is unstable, while for $B = 0.002$ and 0.006 , it is stable. To determine the optimal value of B , we follow [11, 17] and choose $B = B_*$, for which

$$g(\lambda, B_*, \pi) = 0.\tag{2.13}$$

The use of B_* will enforce the strongest damping of grid-scale oscillations, which correspond to the highest-frequency mode $\alpha = \pi$. Substituting (2.10)–(2.11) into (2.13) results in $B_* = 1/256$.

Note that the value of B_* is independent of λ since $\delta(\pi) = 0$. It is instructive, however, to study the stability property of the optimal QL seventh-order scheme for different λ . To this end, we plot the values of $g(\lambda, B_*, \alpha_\ell)$ for $\lambda = 0.25, 0.5, 0.75$, and 1 in Figure 2.2, where one can see that the optimal QL seventh-order scheme becomes unstable if large CFL numbers are used.

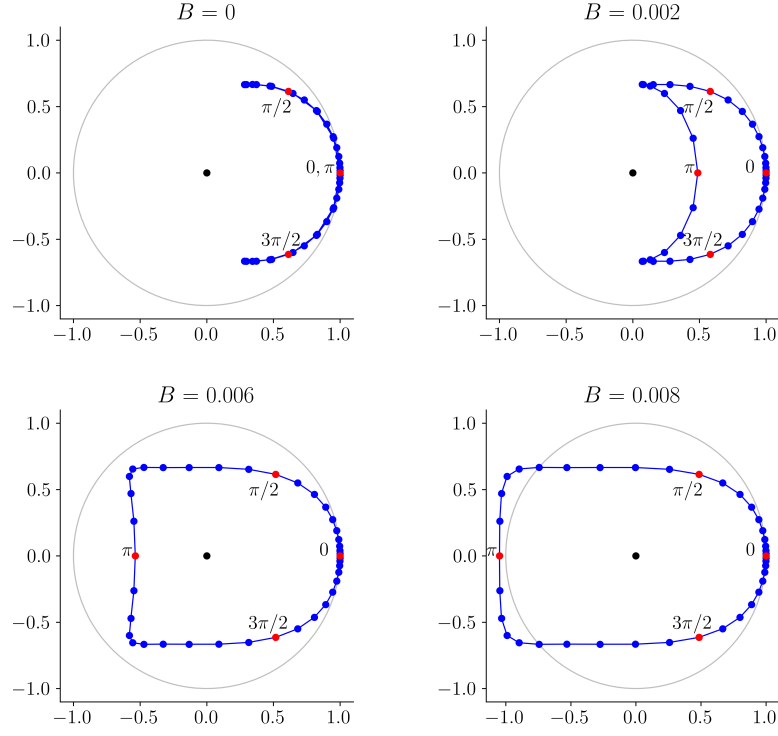


Figure 2.1: QL seventh-order scheme: The values $g = g(0.5, B, \alpha)$ relative to the unit circle in the complex plane for $\alpha = \alpha_\ell$ given by (2.12) and different B .

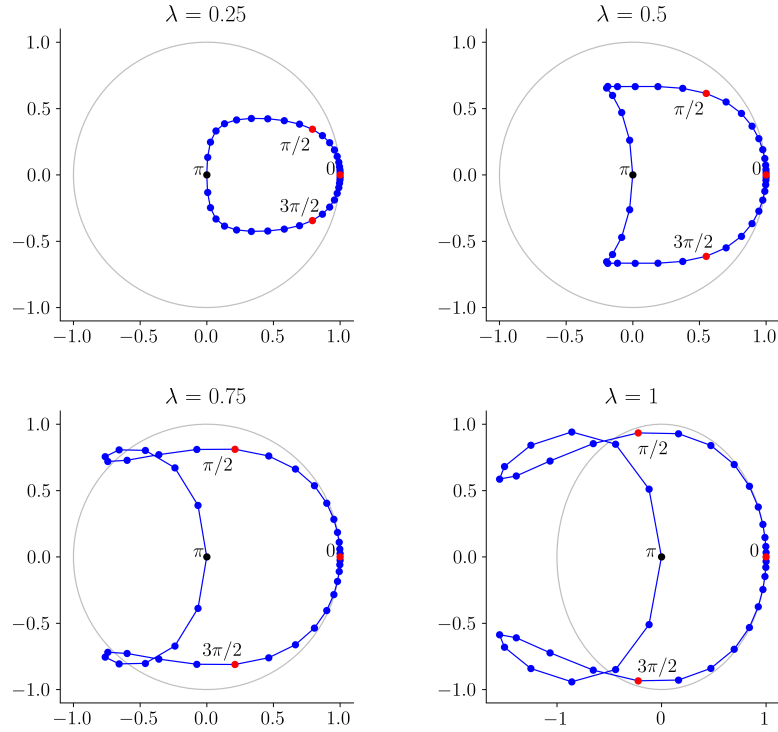


Figure 2.2: QL seventh-order scheme: The values $g(\lambda, B_*, \alpha)$ relative to the unit circle in the complex plane for $\alpha = \alpha_\ell$ given by (2.12) and different λ .

2.3 Von Neumann Stability Analysis of the Quasi-Linear Ninth-Order Schemes

We now follow the approach presented in §2.2 and perform the von Neumann stability analysis for the QL ninth-order scheme, which reads as (2.4)–(2.5) with

$$\begin{aligned}\mathcal{L}_{j+\frac{1}{2}}[U] &= \frac{1}{2520} [2\mathbf{F}(U_{j+5}) - 23\mathbf{F}(U_{j+4}) + 127\mathbf{F}(U_{j+3}) - 473\mathbf{F}(U_{j+2}) + 1627\mathbf{F}(U_{j+1}) \\ &\quad + 1627\mathbf{F}(U_j) - 473\mathbf{F}(U_{j-1}) + 127\mathbf{F}(U_{j-2}) - 23\mathbf{F}(U_{j-3}) + 2\mathbf{F}(U_{j-4})], \\ \omega_{j+\frac{1}{2}}^n &= \frac{3}{2048\lambda^n} [U_{j+5}^n - 9U_{j+4}^n + 36U_{j+3}^n - 84U_{j+2}^n + 126U_{j+1}^n \\ &\quad - 126U_j^n + 84U_{j-1}^n - 36U_{j-2}^n + 9U_{j-3}^n - U_{j-4}^n].\end{aligned}\tag{2.14}$$

We apply these schemes to the linear advection equation (2.8) (once again, in this case, $\lambda \equiv \lambda$), substitute the Fourier mode (2.9) into the scheme (2.4)–(2.5), (2.14), and collect like terms, yielding the amplification factor (2.10) with

$$\begin{aligned}\delta(\alpha) &= -\frac{1}{1260}(e^{5i\alpha} - e^{-5i\alpha}) + \frac{5}{504}(e^{4i\alpha} - e^{-4i\alpha}) - \frac{5}{84}(e^{3i\alpha} - e^{-3i\alpha}) \\ &\quad + \frac{5}{21}(e^{2i\alpha} - e^{-2i\alpha}) - \frac{5}{6}(e^{i\alpha} - e^{-i\alpha}) \\ &= -i \left(\frac{1}{630} \sin(5\alpha) - \frac{5}{252} \sin(4\alpha) + \frac{5}{42} \sin(3\alpha) - \frac{10}{21} \sin(2\alpha) + \frac{5}{3} \sin(\alpha) \right), \\ \Theta(\alpha) &= e^{5i\alpha} + e^{-5i\alpha} - 10(e^{4i\alpha} + e^{-4i\alpha}) + 45(e^{3i\alpha} + e^{-3i\alpha}) - 120(e^{2i\alpha} + e^{-2i\alpha}) \\ &\quad + 210(e^{i\alpha} + e^{-i\alpha}) - 252 \\ &= 2 \cos(5\alpha) - 20 \cos(4\alpha) + 90 \cos(3\alpha) - 240 \cos(2\alpha) + 420 \cos \alpha - 252.\end{aligned}\tag{2.15}$$

In Figure 2.3, we plot the values of g for $\lambda = 0.5$ for different values α defined by (2.12), and $B = 0, 0.0005, 0.0015$, and 0.002 . As one can see, when $B = 0.002$, the scheme is unstable, while for $B = 0.0005$ and 0.0015 , it is stable. To determine the optimal value of B , we substitute (2.10) and (2.15) into (2.13), which results in $B_* = 1/1024$.

We also plot the values of $g(\lambda, B_*, \alpha_\ell)$ for different λ in Figure 2.4, where one can see that the optimal QL ninth-order scheme admits a larger stability region and satisfies the von Neumann condition up to $\lambda = 0.75$.

2.4 1-D Accuracy Test

In this section, we demonstrate that the described 1-D QL schemes achieve seventh and ninth orders of accuracy in capturing smooth solutions. We consider the Euler equations of gas dynamics, which in the 1-D case, read as (1.1) with $\mathbf{U} = (\rho, \rho u, E)^\top$ and $\mathbf{F} = (\rho u, \rho u^2 + p, u(E + p))^\top$. Here, ρ , u , p , and E are the density, velocity, pressure, and total energy, respectively, and the system is closed through the following equations of state (EOS) for ideal gases:

$$p = (\gamma - 1) \left[E - \frac{1}{2} \rho u^2 \right],$$

where the parameter γ represents the specific heat ratio.

We consider the following smooth initial data [9]:

$$u(x, 0) = \sin\left(\frac{\pi x}{5} + \frac{\pi}{4}\right), \quad \rho(x, 0) = \left[\frac{\gamma - 1}{2\sqrt{\gamma}} (u(x, 0) + 10) \right]^{\frac{2}{\gamma-1}}, \quad p(x, 0) = \rho^\gamma(x, 0),$$

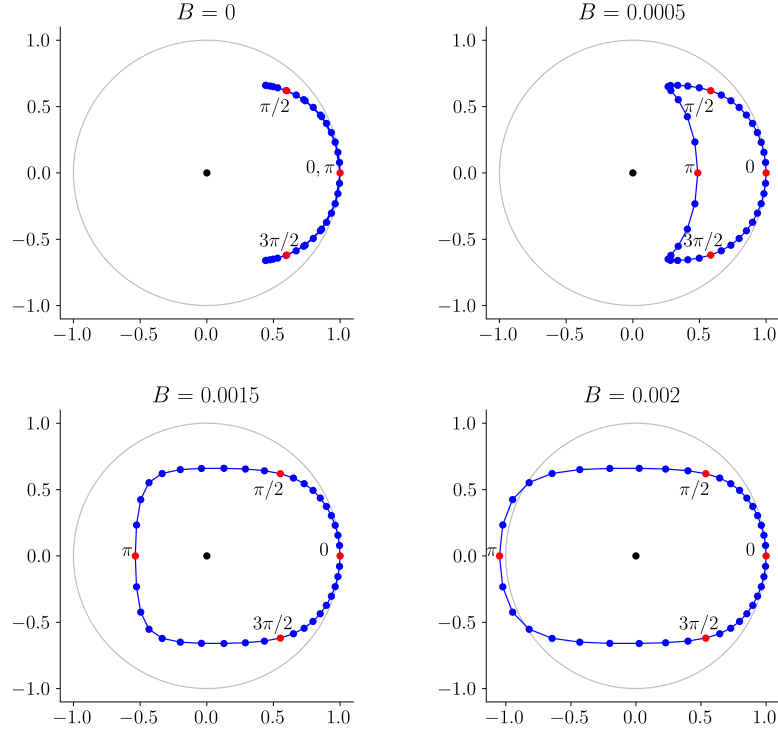


Figure 2.3: QL ninth-order scheme: The values $g = g(0.5, B, \alpha)$ relative to the unit circle in the complex plane for $\alpha = \alpha_\ell$ given by (2.12) and different B .

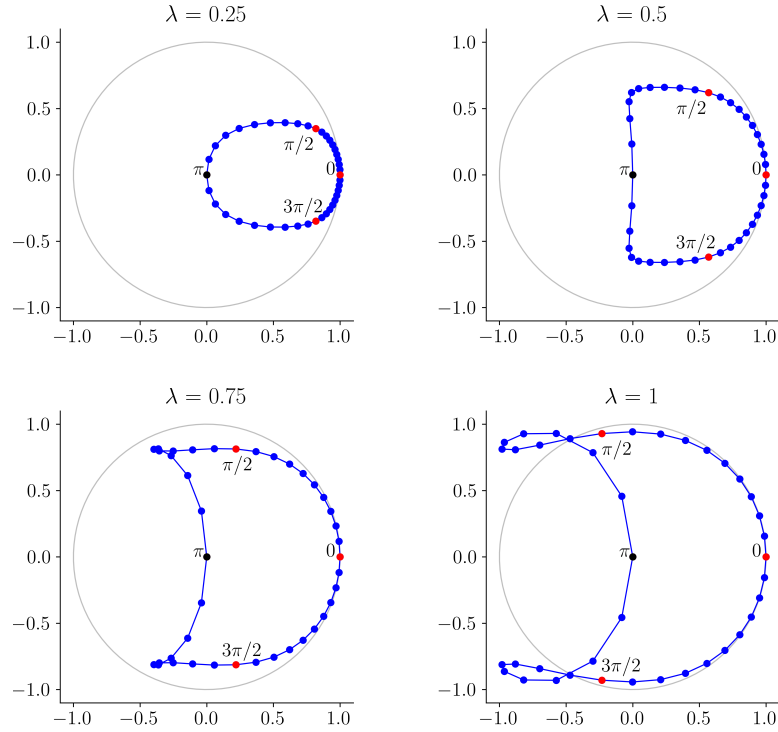


Figure 2.4: QL ninth-order scheme: The values $g(\lambda, B_*, \alpha)$ relative to the unit circle in the complex plane for $\alpha = \alpha_\ell$ given by (2.12) and different λ .

subject to the periodic boundary conditions in the computational domain $[0, 10]$. We compute the numerical solution until the final time $t = 0.1$ on a sequence of uniform meshes with $\Delta x = 1/2, 1/4, 1/8$, and $1/16$.

We then compute the L^1 -errors and estimate the experimental convergence rates for the density ρ using the following Runge formulae, which are based on the solutions computed on the three consecutive uniform grids with the mesh sizes Δx , $2\Delta x$, and $4\Delta x$ and denoted by $(\cdot)^{\Delta x}$, $(\cdot)^{2\Delta x}$, and $(\cdot)^{4\Delta x}$, respectively:

$$\text{Error}(\Delta x) \approx \frac{\delta_{12}^2}{|\delta_{12} - \delta_{24}|}, \quad \text{Rate}(\Delta x) \approx \log_2 \left(\frac{\delta_{24}}{\delta_{12}} \right).$$

Here, $\delta_{12} := \|(\cdot)^{\Delta x} - (\cdot)^{2\Delta x}\|_{L^1}$ and $\delta_{24} := \|(\cdot)^{2\Delta x} - (\cdot)^{4\Delta x}\|_{L^1}$. The obtained results are reported in Table 2.1, where one can clearly see that the expected orders of accuracy are achieved.

Δx	QL Seventh-Order Scheme		QL Ninth-Order Scheme	
	Error	Rate	Error	Rate
1/8	2.28e-08	7.05	4.03e-10	8.99
1/16	1.80e-10	7.02	8.14e-13	8.97

Table 2.1: The L^1 -errors and experimental convergence rates for the QL seventh- and ninth-order schemes.

Remark 2.1 *We stress that in order to achieve the seventh and ninth order of accuracy, we have used smaller time steps with $\Delta t \sim (\Delta x)^{\frac{7}{3}}$ and $\Delta t \sim (\Delta x)^3$ to balance the spatial and temporal errors for the seventh- and ninth-order QL schemes, respectively.*

2.5 1-D Adaptive Schemes

The 1-D adaptive schemes are similar to the scheme introduced in [3, §3], except that the QL fifth-order scheme from [11] is now replaced by either the seventh- or ninth-order QL schemes in the smooth regions, which we use with the optimal values $B = B_*$.

3 Two-Dimensional Schemes

We now extend the 1-D adaptive schemes described in §2 to the 2-D hyperbolic systems of conservation laws (1.2). We assume that the computational domain is covered with uniform cells $I_{j,k} := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$ with $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$ and $y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}} \equiv \Delta y$ centered at (x_j, y_k) with $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $y_k = (y_{k-\frac{1}{2}} + y_{k+\frac{1}{2}})/2$.

3.1 Quasi-Linear r^{th} -Order Schemes

We extend the 1-D QL r^{th} -order schemes described in §2.1 to the 2-D case, which can be done in a dimension-by-dimension manner.

Assuming the point values $U_{j,k}^n \approx U(x_j, y_k, t^n)$ are available, the solution is evolved to the next time

level t^{n+1} according to

$$\begin{aligned} U_{j,k}^I &= U_{j,k}^n - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2},k} - \mathcal{F}_{j-\frac{1}{2},k} \right) - \mu^n \left(\mathcal{G}_{j,k+\frac{1}{2}} - \mathcal{G}_{j,k-\frac{1}{2}} \right), \\ U_{j,k}^{II} &= \frac{3}{4} U_{j,k}^n + \frac{1}{4} \left[U_{j,k}^I - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2},k}^I - \mathcal{F}_{j-\frac{1}{2},k}^I \right) - \mu^n \left(\mathcal{G}_{j,k+\frac{1}{2}}^I - \mathcal{G}_{j,k-\frac{1}{2}}^I \right) \right], \\ U_{j,k}^{n+1} &= \frac{1}{3} U_{j,k}^n + \frac{2}{3} \left[U_{j,k}^{II} - \lambda^n \left(\mathcal{F}_{j+\frac{1}{2},k}^{II} - \mathcal{F}_{j-\frac{1}{2},k}^{II} \right) - \mu^n \left(\mathcal{G}_{j,k+\frac{1}{2}}^{II} - \mathcal{G}_{j,k-\frac{1}{2}}^{II} \right) \right]. \end{aligned} \quad (3.1)$$

where $\lambda^n := \Delta t^n / \Delta x$, $\mu^n := \Delta t^n / \Delta y$, and the numerical fluxes are defined by

$$\begin{aligned} \mathcal{F}_{j+\frac{1}{2},k} &= \mathcal{L}_{j+\frac{1}{2},k}^x[U^n], \quad \mathcal{F}_{j+\frac{1}{2},k}^I = \mathcal{L}_{j+\frac{1}{2},k}^x[U^I], \quad \mathcal{F}_{j+\frac{1}{2},k}^{II} = \mathcal{L}_{j+\frac{1}{2},k}^x[U^{II}] - \omega_{j+\frac{1}{2},k}^{x,n}, \\ \mathcal{G}_{j,k+\frac{1}{2}} &= \mathcal{L}_{j,k+\frac{1}{2}}^y[U^n], \quad \mathcal{G}_{j,k+\frac{1}{2}}^I = \mathcal{L}_{j,k+\frac{1}{2}}^y[U^I], \quad \mathcal{G}_{j,k+\frac{1}{2}}^{II} = \mathcal{L}_{j,k+\frac{1}{2}}^y[U^{II}] - \omega_{j,k+\frac{1}{2}}^{y,n}. \end{aligned} \quad (3.2)$$

Here,

$$\begin{aligned} \mathcal{L}_{j+\frac{1}{2},k}^x[U] &:= \sum_{i=1-\ell}^{\ell} b_i \mathbf{F}(U_{j+i,k}), \quad \omega_{j+\frac{1}{2},k}^{x,n} := \frac{3B_*}{2\lambda^n} \sum_{i=1-\ell}^{\ell} (-1)^{i+\ell} \binom{2\ell-1}{i+\ell-1} \mathbf{F}(U_{j+i,k}), \\ \mathcal{L}_{j,k+\frac{1}{2}}^y[U] &:= \sum_{i=1-\ell}^{\ell} b_i \mathbf{G}(U_{j,k+i}), \quad \omega_{j,k+\frac{1}{2}}^{y,n} := \frac{3B_*}{2\mu^n} \sum_{i=1-\ell}^{\ell} (-1)^{i+\ell} \binom{2\ell-1}{i+\ell-1} \mathbf{G}(U_{j,k+i}), \end{aligned}$$

where b_i are given by (2.6) and the optimal value of B_* is the same as the 1-D case, namely, $B_* = 1/2^{r+1}$.

3.2 2-D Accuracy Test

In this section, we demonstrate that the developed 2-D QL schemes achieve the seventh and ninth orders of accuracy in capturing smooth solutions. As in the 1-D case, we focus on the Euler equations of gas dynamics, whose 2-D version reads as (1.2) with

$$\mathbf{U} = (\rho, \rho u, \rho v, E)^\top, \quad \mathbf{F} = (\rho u, \rho u^2 + p, \rho uv, u(E + p))^\top, \quad \mathbf{G} = (\rho v, \rho uv, \rho v^2 + p, v(E + p))^\top. \quad (3.3)$$

Here, v is the y -velocity and the other variables are the same as in the 1-D case. The system (1.2), (3.3) is closed through the following EOS for ideal gases:

$$p = (\gamma - 1) \left[E - \frac{\rho}{2} (u^2 + v^2) \right]. \quad (3.4)$$

We consider the following smooth initial data [2, 20]:

$$\begin{aligned} \rho(x, y, 0) &= \left(1 - \frac{(\gamma - 1)\kappa^2}{2\gamma} \right)^{\frac{1}{\gamma-1}}, \quad p(x, y, 0) = \rho^\gamma(x, y, 0), \\ u(x, y, 0) &= 1 - \kappa y, \quad v(x, y, 0) = 1 + \kappa x, \quad \kappa = \frac{5}{2\pi} e^{\frac{1-x^2-y^2}{2}}, \end{aligned}$$

subject to the periodic boundary conditions in the computational domain $[-10, 10] \times [-10, 10]$. The exact solution of this initial value problem is given by $\mathbf{U}(x, y, t) = \mathbf{U}(x - t, y - t, 0)$.

We compute the numerical solution until the final time $t = 0.1$ on a sequence of uniform meshes with $\Delta x = \Delta y = 1/10, 1/20$, and $1/40$. The time steps are selected to be $\Delta t \sim (\Delta x)^{\frac{7}{3}}$ and $\Delta t \sim (\Delta x)^3$ for the seventh- and ninth-order schemes, respectively. We then measure the L^1 -errors and the corresponding experimental convergence rates for the density ρ . The obtained results are presented in Table 3.1, where one can see that the expected orders of accuracy have been achieved.

$\Delta x = \Delta y$	QL Seventh-Order Scheme		QL Ninth-Order Scheme	
	Error	Rate	Error	Rate
1/10	2.01e-05	—	7.09e-06	—
1/20	1.57e-07	7.00	1.43e-08	8.96
1/40	1.27e-09	6.95	2.88e-11	8.95

Table 3.1: The L^1 -errors and experimental convergence rates for the QL seventh- and ninth-order schemes.

3.3 2-D Adaptive Schemes

As in the 1-D case, the 2-D adaptive schemes are similar to the one introduced in [3, §4.3], except that the QL fifth-order schemes from [11] are replaced by either the seventh- or ninth-order QL schemes in the smooth regions. These schemes read as (3.1)–(3.2) with

$$\begin{aligned}
\mathcal{L}_{j+\frac{1}{2},k}^x[U] &= \frac{1}{840} \left[-3F(U_{j+4,k}) + 29F(U_{j+3,k}) - 139F(U_{j+2,k}) + 533F(U_{j+1,k}) \right. \\
&\quad \left. + 533F(U_{j,k}) - 139F(U_{j-1,k}) + 29F(U_{j-2,k}) - 3F(U_{j-3,k}) \right], \\
\omega_{j+\frac{1}{2},k}^{x,n} &= \frac{3}{512\lambda^n} \left[-U_{j+4,k}^n + 7U_{j+3,k}^n - 21U_{j+2,k}^n + 35U_{j+1,k}^n - 35U_{j,k}^n + 21U_{j-1,k}^n - 7U_{j-2,k}^n + U_{j-3,k}^n \right], \\
\mathcal{L}_{j,k+\frac{1}{2}}^y[U] &= \frac{1}{840} \left[-3G(U_{j,k+4}) + 29G(U_{j,k+3}) - 139G(U_{j,k+2}) + 533G(U_{j,k+1}) \right. \\
&\quad \left. + 533G(U_{j,k}) - 139G(U_{j,k-1}) + 29G(U_{j,k-2}) - 3G(U_{j,k-3}) \right], \\
\omega_{j,k+\frac{1}{2}}^{y,n} &= \frac{3}{512\mu^n} \left[-U_{j,k+4}^n + 7U_{j,k+3}^n - 21U_{j,k+2}^n + 35U_{j,k+1}^n - 35U_{j,k}^n + 21U_{j,k-1}^n - 7U_{j,k-2}^n + U_{j,k-3}^n \right],
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}_{j+\frac{1}{2},k}^x[U] &= \frac{1}{2520} \left[2F(U_{j+5,k}) - 23F(U_{j+4,k}) + 127F(U_{j+3,k}) - 473F(U_{j+2,k}) + 1627F(U_{j+1,k}) \right. \\
&\quad \left. + 1627F(U_{j,k}) - 473F(U_{j-1,k}) + 127F(U_{j-2,k}) - 23F(U_{j-3,k}) + 2F(U_{j-4,k}) \right], \\
\omega_{j+\frac{1}{2},k}^{x,n} &= \frac{3}{2048\lambda^n} \left[U_{j+5,k}^n - 9U_{j+4,k}^n + 36U_{j+3,k}^n - 84U_{j+2,k}^n + 126U_{j+1,k}^n \right. \\
&\quad \left. - 126U_{j,k}^n + 84U_{j-1,k}^n - 36U_{j-2,k}^n + 9U_{j-3,k}^n - U_{j-4,k}^n \right], \\
\mathcal{L}_{j,k+\frac{1}{2}}^y[U] &= \frac{1}{2520} \left[2G(U_{j,k+5}) - 23G(U_{j,k+4}) + 127G(U_{j,k+3}) - 473G(U_{j,k+2}) + 1627G(U_{j,k+1}) \right. \\
&\quad \left. + 1627G(U_{j,k}) - 473G(U_{j,k-1}) + 127G(U_{j,k-2}) - 23G(U_{j,k-3}) + 2G(U_{j,k-4}) \right], \\
\omega_{j,k+\frac{1}{2}}^{y,n} &= \frac{3}{2048\mu^n} \left[U_{j,k+5}^n - 9U_{j,k+4}^n + 36U_{j,k+3}^n - 84U_{j,k+2}^n + 126U_{j,k+1}^n \right. \\
&\quad \left. - 126U_{j,k}^n + 84U_{j,k-1}^n - 36U_{j,k-2}^n + 9U_{j,k-3}^n - U_{j,k-4}^n \right],
\end{aligned}$$

for the seventh- and ninth-order schemes, respectively.

4 Numerical Examples

In this section, we apply the developed adaptive schemes to both the 1-D and 2-D Euler equations of gas dynamics and compare their performance with that of the adaptive schemes from [3]. We take $\gamma = 1.4$ in Examples 1–5 and $\gamma = 5/3$ in Example 6. In all of the examples, we use the CFL number 0.4.

For the sake of brevity, the adaptive schemes which are based on the QL fifth-, seventh-, and ninth-order schemes will be referred to as the 5-Order, 7-Order, and 9-Order schemes, respectively.

In the examples below, we specify the values of the adaption constants C_1 and C_2 . For their specific definitions, we refer the reader to [3], where details on the adaptive algorithms can be found.

All of the numerical experiments have been conducted on a workstation equipped with an Intel(R) Core(TM) i9-13900H CPU at 2.6 GHz and 32 GB of RAM. The simulations were conducted in FORTRAN using GCC version 14.2.0 compiler suite. The reported CPU times were averaged over thirty independent runs to ensure reproducibility and minimize variability due to system processes.

4.1 1-D Examples

In the 1-D examples, the adaptive solutions will be plotted along with the reference solutions, which will be computed using the second-order LDCU scheme from [6] and implemented with the Minmod2 limiter on much finer meshes.

Example 1—Titarev-Toro Problem. In the first example taken from [23] (see also [22, 25]), we consider the shock-entropy wave interaction problem with the following initial conditions:

$$(\rho, u, p)\Big|_{(x,0)} = \begin{cases} (1.51695, 0.523346, 1.805), & x < -4.5, \\ (1 + 0.1 \sin(20x), 0, 1), & x > -4.5, \end{cases}$$

prescribed in the computational domain $[-5, 5]$ subject to the free boundary conditions. In this initial boundary value problem, a forward-facing shock wave of Mach number 1.1 interacts with high-frequency density perturbations: as the shock wave moves, the perturbations spread ahead.

We compute the numerical solutions by the 5-Order, 7-Order, and 9-Order schemes (with the adaption constants $C_1 = 0.02$ and $C_2 = 0.3$) until the final time $t = 5$ on a uniform mesh with $\Delta x = 1/60$. The obtained numerical results are presented in Figure 4.1 along with the reference solution computed on a much finer mesh with $\Delta x = 1/1600$, where one can see that the 7-Order and 9-Order schemes are very close but both are more accurate than the 5-Order one.

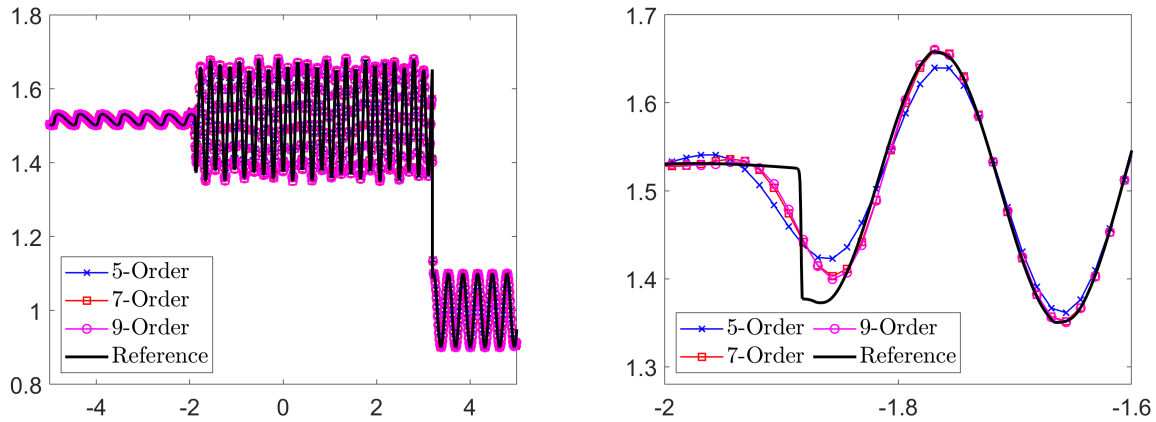


Figure 4.1: Example 1: Density ρ computed by the 5-Order, 7-Order, and 9-Order schemes (left) and zoom at $x \in [-2, -1.6]$ (right).

Example 2—Blast Wave Problem. In the second 1-D example, we consider a strong shocks interaction problem from [26] with the initial data,

$$(\rho, u, p)\Big|_{(x,0)} = \begin{cases} (1, 0, 1000), & x < 0.1, \\ (1, 0, 0.01), & 0.1 \leq x \leq 0.9, \\ (1, 0, 100), & x > 0.9, \end{cases}$$

prescribed in the computational domain $[0, 1]$ subject to the solid wall boundary conditions.

We compute the numerical solution until the final time $t = 5$ by the 5-Order, 7-Order, and 9-Order schemes (with the adaption constants $C_1 = 0.002$ and $C_2 = 0.3$) on a uniform mesh with $\Delta x = 1/400$ and present the obtained numerical results in Figure 4.2 together with the reference solution computed on a much finer mesh with $\Delta x = 1/4000$. It can be observed that all of the studied adaptive schemes are capable of achieving a superb resolution of the contact wave located at about $x = 0.6$. The obtained results are quite similar, but the main goal of this example is to show that the new adaptive schemes do not produce any spurious oscillations while capturing the numerical solutions of this challenging benchmark.

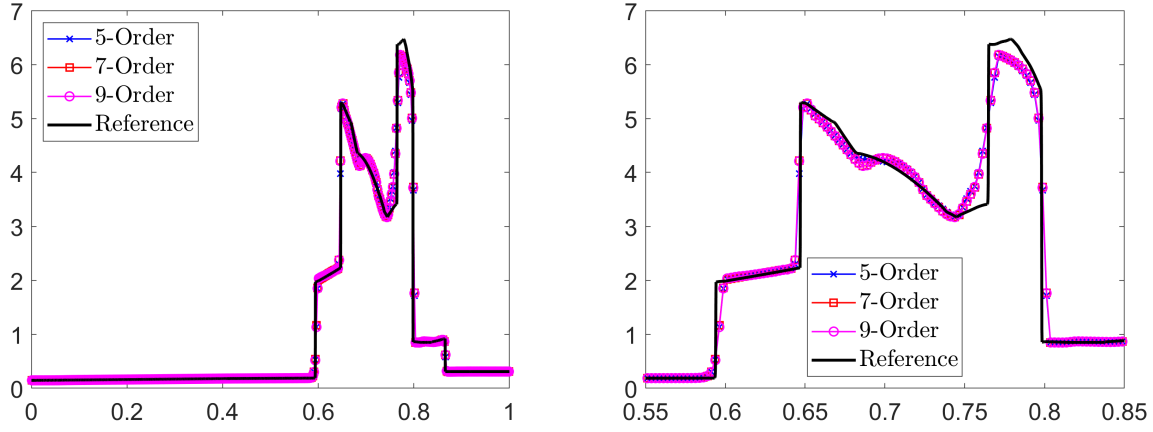


Figure 4.2: Example 2: Density ρ computed by the 5-Order, 7-Order, and 9-Order schemes (left) and zoom at $x \in [0.55, 0.85]$.

4.2 2-D Examples

In the 2-D examples, we will not only show the obtained solutions, but will also plot the areas, which were identified as “rough” at the last time step. The contact discontinuities areas will be marked in red, while the rest of the “rough” areas will be indicated in black.

Example 3—2-D Riemann Problem (Configuration 3). In this example taken from [12], the initial conditions,

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (1.5, 0, 0, 1.5), & x > 1, y > 1, \\ (0.5323, 1.206, 0, 0.3), & x < 1, y > 1, \\ (0.138, 1.206, 1.206, 0.029), & x < 1, y < 1, \\ (0.5323, 0, 1.206, 0.3), & x > 1, y < 1, \end{cases}$$

are prescribed in the computational domain $[0, 1.2] \times [0, 1.2]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t = 1$ by the 5-Order (with the adaption constants $C_1 = 0.08$ and $C_2 = 0.2$), 7-Order (with the adaption constants $C_1 = 0.08$ and $C_2 = 0.15$), and 9-Order (with the adaption constants $C_1 = 0.1$ and $C_2 = 0.2$) schemes on a uniform mesh with $\Delta x = \Delta y = 3/2500$. The obtained results are plotted in Figure 4.3 (top row), where one can see that the 7-Order and 9-Order schemes slightly outperform the 5-Order one in capturing a sideband instability of the jet in the zones of strong along-jet velocity shear and the instability along the jets neck. The detected “rough” areas for the 5-Order, 7-Order, and 9-Order schemes are plotted in Figure 4.3 (bottom row), where one can see that the overcompressive and Minmod2 limiters are implemented only in a small part of the computational domain.

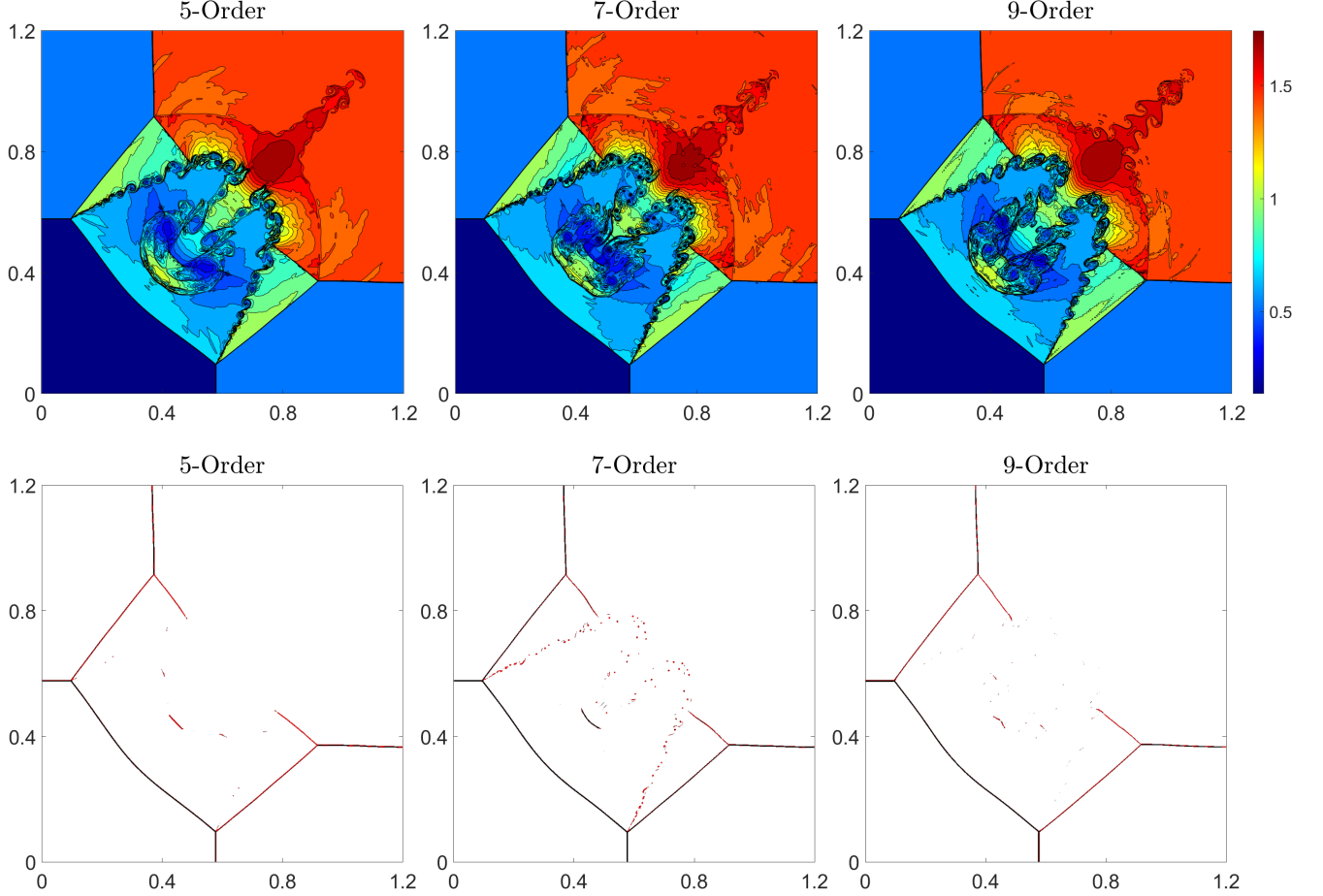


Figure 4.3: Example 3: Top row: Density ρ computed by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes. Bottom row: The “rough” areas detected by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes at the final time step.

Example 4—2-D Riemann Problem (Configuration 6). In this example also taken from [12], the initial conditions,

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (1, 0.75, -0.5, 1), & x > 0.5, y > 0.5, \\ (2, 0.75, 0.5, 1), & x < 0.5, y > 0.5, \\ (1, -0.75, 0.5, 1), & x < 0.5, y < 0.5, \\ (3, -0.75, -0.5, 1), & x > 0.5, y < 0.5, \end{cases}$$

are prescribed in the computational domain $[0, 1] \times [0, 1]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t = 1$ by the 5-Order, 7-Order, and 9-Order (with the adaption constants $C_1 = 0.06$ and $C_2 = 0.004$) schemes on a uniform mesh with $\Delta x = \Delta y = 1/600$. The obtained results are plotted in Figure 4.4 (top row), where one can see that the 7-Order and 9-Order schemes outperform the 5-Order one in capturing much more complicated vortex structures and the 9-Order scheme achieves the highest resolution of small solution structures. At the same time, the “rough” areas identified by the 5-Order, 7-Order, and 9-Order schemes are quite similar; see Figure 4.4 (bottom row).

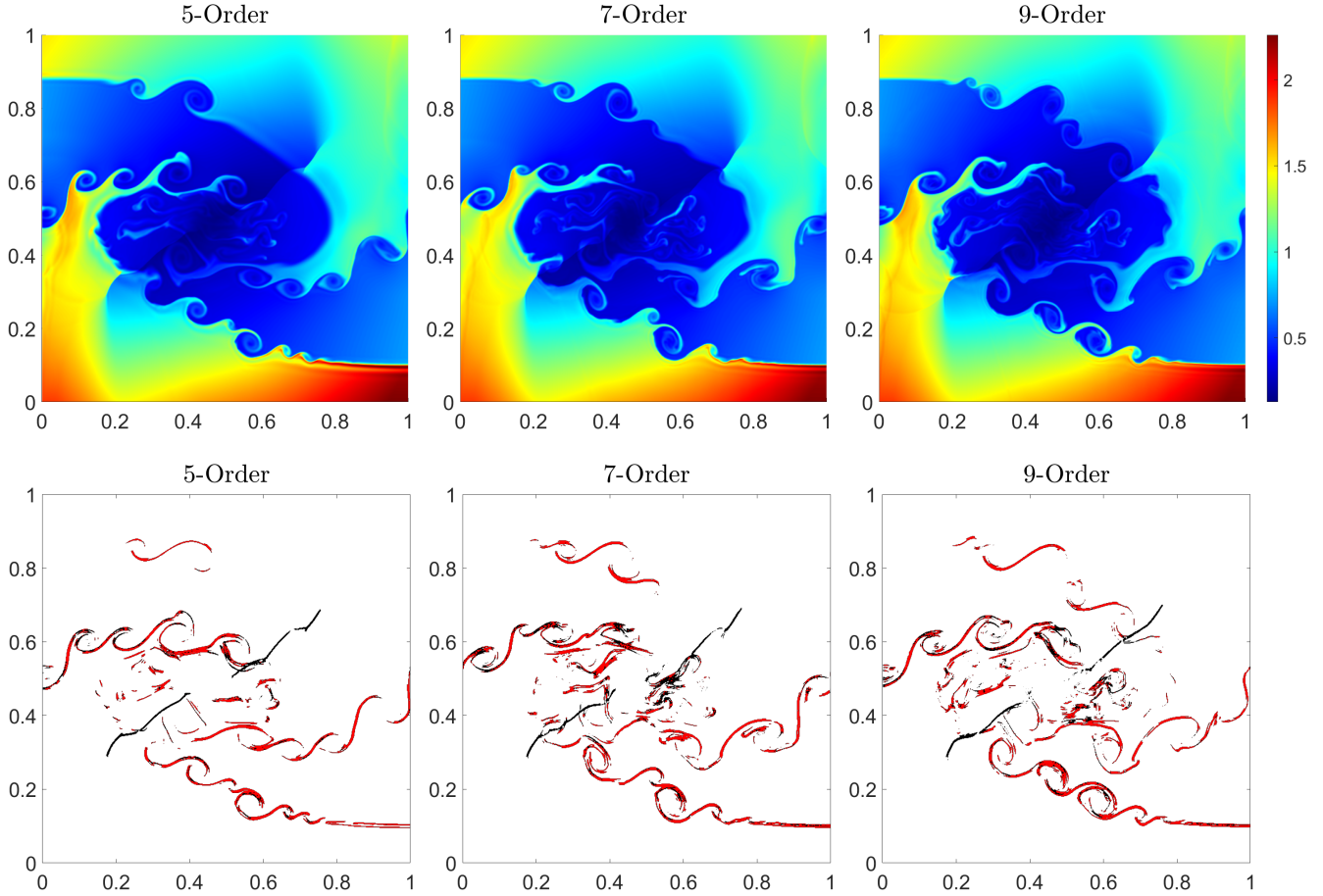


Figure 4.4: Example 4: Top row: Density ρ computed by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes. Bottom row: The “rough” areas detected by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes at the final time step.

Example 5—Implosion Problem. In this example, we consider the implosion problem taken from [15]. The initial conditions,

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (0.125, 0, 0, 0.14), & |x| + |y| < 0.15, \\ (1, 0, 0, 1), & \text{otherwise,} \end{cases}$$

are prescribed in the computational domain $[0, 0.5] \times [0, 0.5]$ with the solid wall boundary conditions. In this setting, a jet forms near the origin and propagates along the diagonal $y = x$ direction. The position of the jet can serve as an indicator of the amount of numerical dissipation present in the studied numerical scheme as schemes containing larger numerical dissipation may not resolve the jet at all or the jet propagation may slow down by excessive numerical diffusion.

We compute the numerical solution until the final time $t = 2.5$ by the 5-Order, 7-Order, and 9-Order (with the adaption constants $C_1 = 0.05$ and $C_2 = 0.1$) schemes on a uniform mesh with $\Delta x = \Delta y = 7/10000$. We present the obtained numerical results in Figure 4.5 (top row), where one can see that the jet propagates further in the diagonal direction in the numerical result computed by the 7-Order and 9-Order schemes, which demonstrates that the use of higher-order schemes in the smooth areas helps to reduce numerical dissipation. At the same time, the “rough” areas detected by the 5-Order, 7-Order, and 9-Order schemes, which are plotted in Figure 4.5 (bottom row), show that nonlinear limiters are implemented only in a very small part of the computational domain.

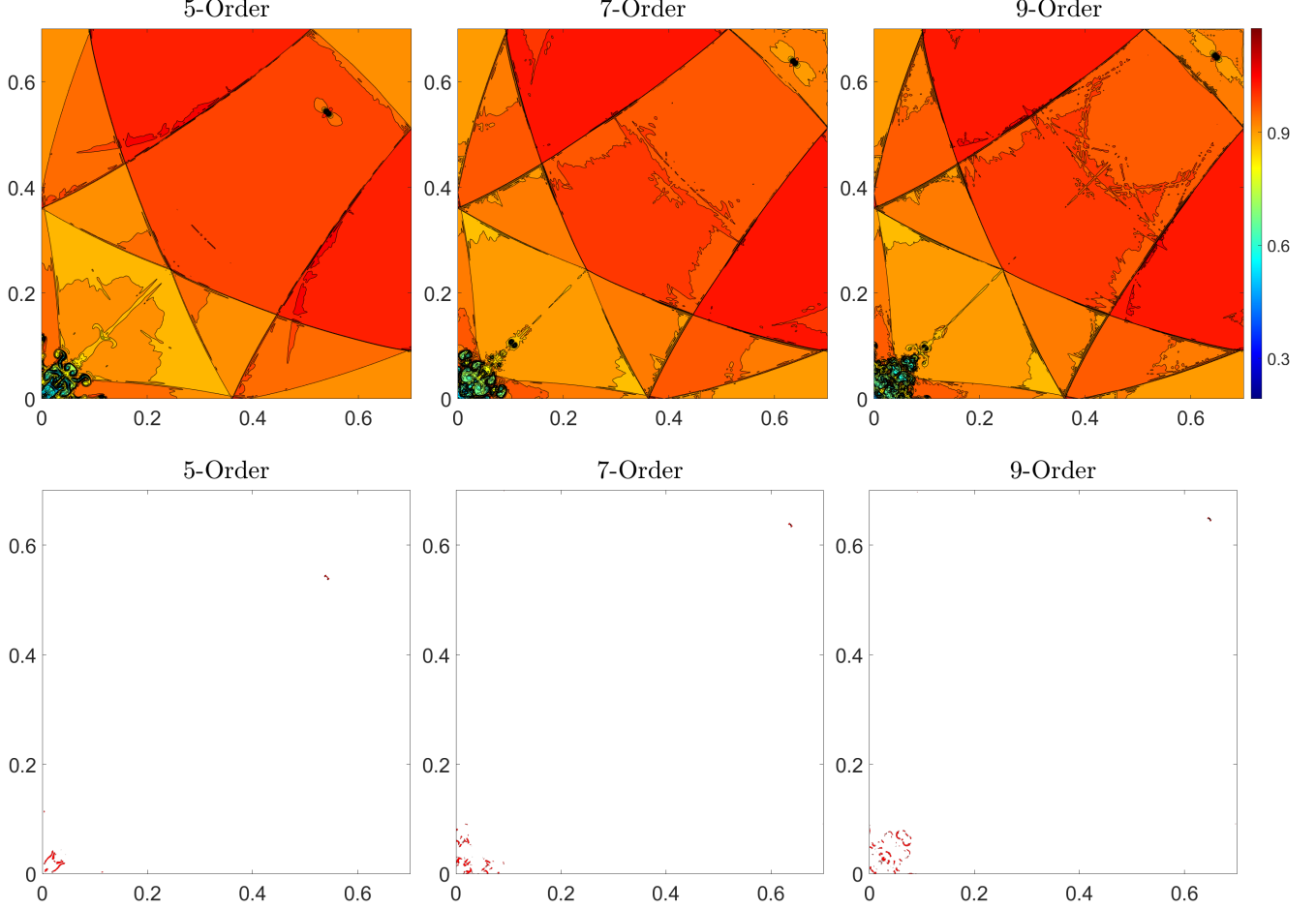


Figure 4.5: Example 5: Top row: Density ρ computed by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes. Bottom row: The “rough” areas detected by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes at the final time step.

Next, we measure the CPU times consumed by the 5-Order, 7-Order, and 9-Order schemes. The obtained results show that the CPU time consumed by the 7-Order and 9-Order schemes are about 7.0% and 11.7% larger than the CPU time consumed by the 5-Order scheme, respectively. This demonstrates that the proposed adaptive algorithms based on new QL seventh-order and ninth-order schemes achieve higher resolution compared with their counterpart based on the QL fifth-order scheme, while consuming almost the same CPU times.

Example 6—RT Instability. In the last example taken from [19], we investigate the RT instability, which is a physical phenomenon occurring when a layer of heavier fluid is placed on top of a layer of

lighter fluid.

We consider the 2-D Euler equations of gas dynamics (1.2), (3.3)–(3.4) with the gravitational source terms acting in the positive direction of the y -axis:

$$\begin{aligned}\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= \rho, \\ E_t + [u(E + p)]_x + [v(E + p)]_y &= \rho v,\end{aligned}$$

and take the following initial conditions:

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (2, 0, -0.025c \cos(8\pi x), 2y + 1), & y < 0.5, \\ (1, 0, -0.025c \cos(8\pi x), y + 1.5), & \text{otherwise,} \end{cases}$$

where $c := \sqrt{\gamma p / \rho}$ is the speed of sound. The solid wall boundary conditions are imposed at $x = 0$ and $x = 0.25$, and the following Dirichlet boundary conditions are prescribed at the top and bottom boundaries:

$$(\rho, u, v, p)(x, 1, t) = (1, 0, 0, 2.5), \quad (\rho, u, v, p)(x, 0, t) = (2, 0, 0, 1).$$

We compute the numerical solution until the final time $t = 2.95$ by the 5-Order (with the adaption constants $\mathbf{C}_1 = 0.08$ and $\mathbf{C}_2 = 0.008$), 7-Order (with the adaption constants $\mathbf{C}_1 = 0.1$ and $\mathbf{C}_2 = 0.007$), and 9-Order (with the adaption constants $\mathbf{C}_1 = 0.12$ and $\mathbf{C}_2 = 0.009$) schemes on a uniform mesh with $\Delta x = \Delta y = 1/1024$ in the computational domain $[0, 0.25] \times [0, 1]$. The obtained numerical results are presented in the top rows of Figures 4.6 and 4.7 at times $t = 1.95$ and 2.95 , respectively. The “rough” areas detected by the studied schemes at the same times are plotted in the bottom rows of Figures 4.6 and 4.7. As one can see, the 7-Order and 9-Order schemes resolve more small solution structures: this indicates that these schemes contain a smaller amount of numerical dissipation compared with their 5-Order counterpart.

Declarations:

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Conflicts of interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data and software availability. The data that support the findings of this study and FORTRAN codes developed by the authors and used to obtain all of the presented numerical results are available from the corresponding author upon reasonable request.

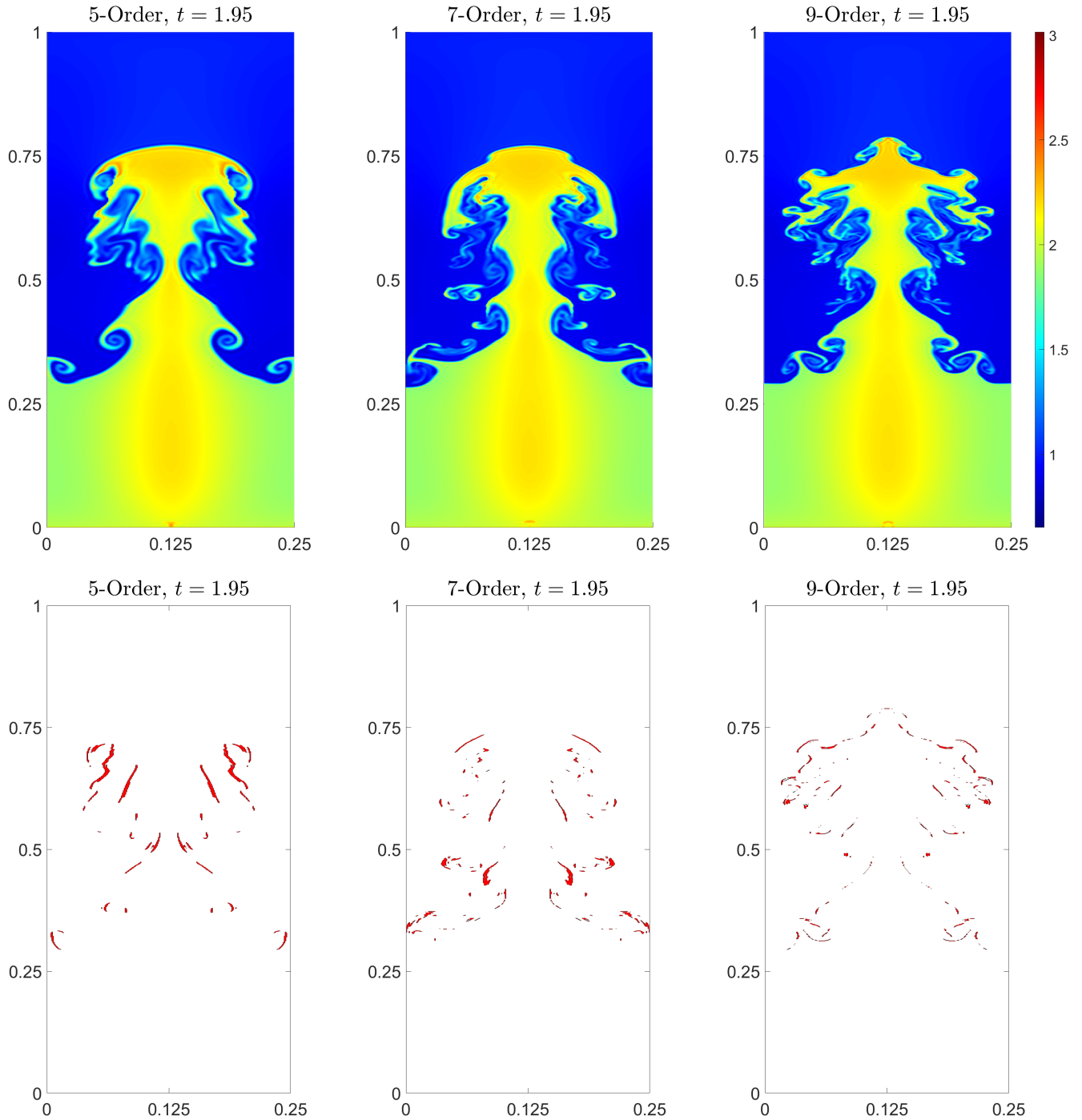


Figure 4.6: Example 6: Top row: Density ρ computed by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes. Bottom row: The “rough” areas detected by the 5-Order (left), 7-Order (middle), and 9-Order (right) schemes at $t = 1.95$.

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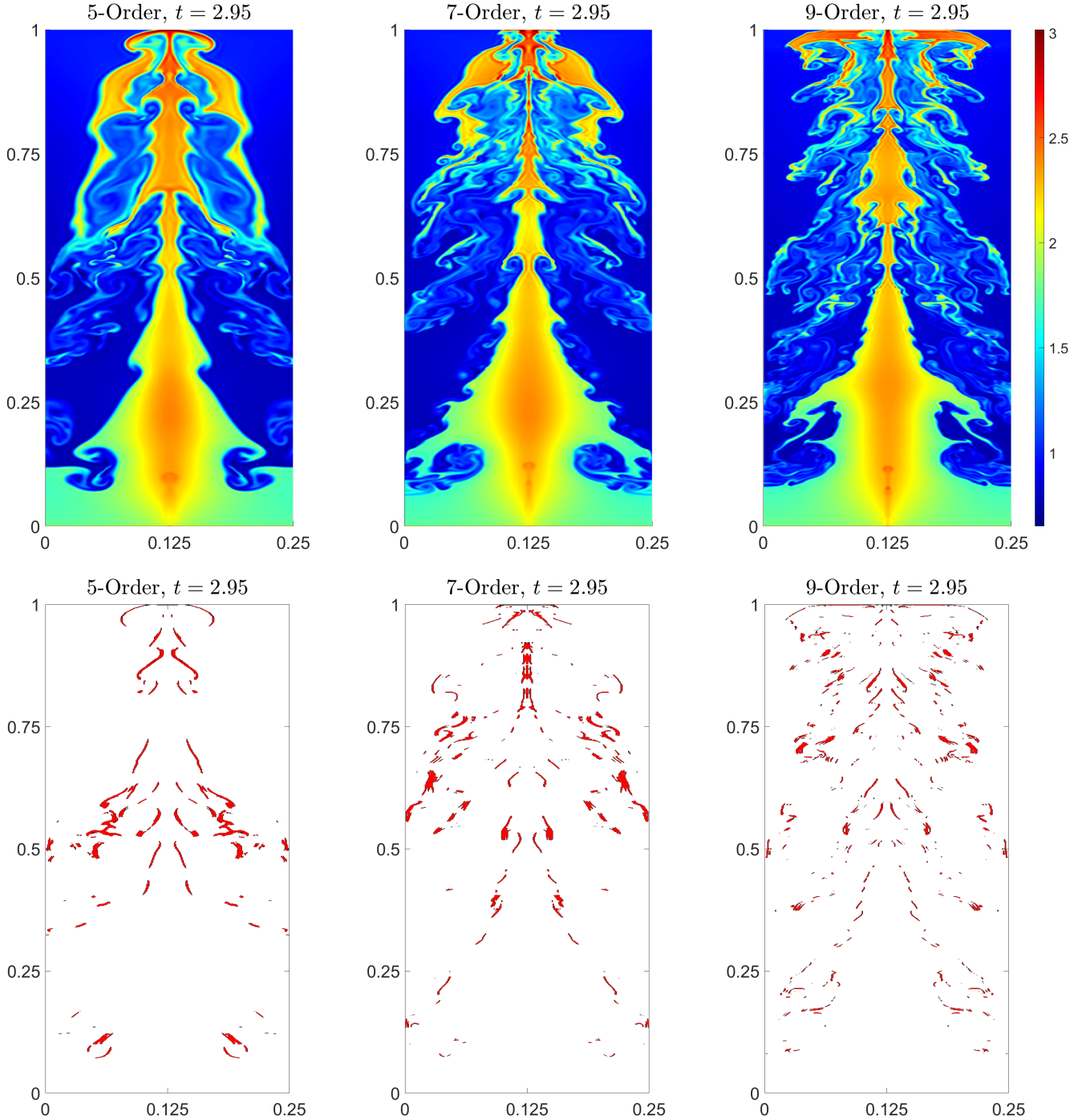


Figure 4.7: Example 6: Same as in Figure 4.6, but at the final time $t = 2.95$.

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