

New Scheme Adaption Strategy for Hyperbolic Conservation Laws

Shaoshuai Chu, Michael Herty, and Alexander Kurganov

Abstract We introduce a new scheme adaption strategy for one- and two-dimensional hyperbolic systems of conservation laws. The proposed approach builds upon the adaptive framework introduced in [S. CHU, A. KURGANOV, AND I. MENSHOV, Appl. Numer. Math., 209 (2025), pp. 155–170], where we first employed the smoothness indicator from [R. LÖHNER, Comput. Methods. Appl. Mech. Eng., 61 (1987), pp. 323–338] to automatically detect “rough” and smooth parts of the computed solution, and then used different limiters in the detected regions. This adaptive strategy was based on a threshold needed to sharply separate “rough” and smooth regions.

In this paper, we propose a different adaption strategy. We use SBM-type limiters and vary one of the limiting parameters continuously to allow a smooth transition between the “rough” and smooth areas. This way, compressive and overcompressive limiters are activated in the shock and contact wave vicinities only, while we gradually switch to dissipative limiters in the smooth regions. A series of one- and two-dimensional numerical tests for the Euler equations of gas dynamics demonstrates that the new scheme adaption strategy leads to a higher resolution and reduced numerical dissipation.

Keywords Scheme adaption · Smoothness indicator · Overcompressive, compressive, and dissipative limiters · Euler equations of gas dynamics

Shaoshuai Chu

Department of Mathematics, RWTH Aachen University, 52056 Aachen, Germany, e-mail: chu@igpm.rwth-aachen.de

Michael Herty

Department of Mathematics, RWTH Aachen University, 52056 Aachen, Germany; Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield 0028, South Africa, e-mail: herty@igpm.rwth-aachen.de

Alexander Kurganov

Department of Mathematics and Shenzhen International Center for Mathematics, Southern University of Science and Technology, Shenzhen, 518055, China, e-mail: alexander@sustech.edu.cn

1 Introduction

This paper focuses on the development of a new scheme adaption strategy for hyperbolic systems of conservation laws, which, in the one-dimensional (1-D) and two-dimensional (2-D) cases, read as

$$U_t + F(U)_x = 0, \quad (1)$$

and

$$U_t + F(U)_x + G(U)_y = 0, \quad (2)$$

respectively. Here, x and y are spatial variables, t is time, $U \in \mathbb{R}^d$ is the vector of unknowns, and $F, G : \mathbb{R}^d \rightarrow \mathbb{R}^d$ are nonlinear flux functions.

It is well-known that, even for smooth initial data, solutions of nonlinear hyperbolic systems may develop complex structures, including shocks, rarefactions, and contact discontinuities. This makes the construction of accurate and robust shock-capturing methods a challenging task. In this work, we restrict our consideration to adaptive methods, which automatically detect “rough” parts of the computed solution and apply different numerical procedures there and in the smooth regions. In particular, we build upon the adaptive strategy introduced in [3], where the low-dissipation central-upwind (LDCU) numerical fluxes from [4] were combined with a piecewise linear reconstruction procedure. The slopes of the linear pieces were computed adaptively. To this end, “rough” regions were detected with the help of the smoothness indicator (SI) proposed in [10]. After that, the two-parameter family of the SBM-type limiters from [9] were used with the parameters corresponding to either an overcompressive limiter (in the “rough” areas) or a dissipative one (in the “smooth” areas). The latter was necessary to suppress artificial kinks, staircase-like structures, and spurious jump discontinuities that may contaminate smooth solution profiles if they are captured using overcompressive limiters; see [9]. Since the limiters utilized in [3] were quite sharp (even the dissipative Minmod2 limiter), they were implemented in local characteristic variables using a local characteristic decomposition.

The main objective of the present paper is to improve the adaptive schemes introduced in [3] by modifying the scheme adaption procedure. In the conventional approach (also used in [3]), the “rough” areas are identified as a group of cells (or cell interfaces), in which the values of the SI are above a prescribed threshold. After that, a decision on which sets of parameters in the SBM-type limiter should be used in “rough” and “smooth” areas is made: this completes the design of a scheme adaption method.

An obvious drawback of such an approach is the sharp separation between the “rough” and “smooth” regions, which often leads to undesired numerical artifacts. If the threshold is chosen too small, the overcompressive limiter is activated in excessively large parts of the computational domain, which may result in artificial kinks, staircase-like structures, and spurious jump discontinuities appearing in the areas where the exact solution is smooth. On the other hand, if the threshold is chosen too large, the dissipative limiter is used in excessively large parts of the computational domain, and captured discontinuities, especially contact ones, may be oversmeared. To overcome these drawbacks, we introduce transition layers between the “rough” and “smooth” areas by smoothly switching the parameters in the SBM-type limiters. This allows us

to keep using very sharp, overcompressive limiters right at the discontinuities and their immediate neighborhood, while switching to compressive and dissipative limiters in the transition layers (this helps to keep the discontinuities much better resolved without risking smooth solution structures to be affected), and then using dissipative limiters inside the “smooth” regions.

The rest of the paper is organized as follows. In §2, we briefly review the 1-D LDCU scheme from [4] and then introduce the new scheme adaption strategy. In §3, we extend the proposed scheme adaption strategy to the 2-D case. In §4, we apply the developed methods to a series of 1-D and 2-D test problems for the Euler equations of gas dynamics and demonstrate that the new scheme adaption strategies lead to higher resolution compared with the resolution achieved by their counterparts from [3].

2 One-Dimensional Scheme Adaption Algorithm

In this section, we consider the 1-D conservation law (1) and describe the proposed adaptive algorithm. We will first briefly review the 1-D LDCU scheme from [4] as well as the SBM-type limiters (§2.1) and then introduce the new 1-D adaptive scheme (§2.2).

2.1 1-D Low-Dissipation Central-Upwind (LDCU) Scheme

Assume that the computational domain is partitioned into uniform cells $I_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ of size $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$, centered at $x_j := (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$. We denote by $\bar{U}_j(t)$ the cell averages of $U(\cdot, t)$ over I_j , namely,

$$\bar{U}_j(t) \approx \frac{1}{\Delta x} \int_{I_j} U(x, t) dx,$$

and assume that, at a given time level $t \geq 0$, the values $\{\bar{U}_j(t)\}$ are available. Note that most of the indexed quantities depend on t , but this dependence will be suppressed hereafter for the sake of brevity.

In the semi-discrete LDCU scheme, the cell averages are evolved by solving the ODE system

$$\frac{d\bar{U}_j}{dt} = -\frac{\mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}}}{\Delta x}, \quad (3)$$

where $\mathcal{F}_{j+\frac{1}{2}} = \mathcal{F}(U_{j+\frac{1}{2}}^-, U_{j+\frac{1}{2}}^+)$ is the LDCU numerical flux. In [4], the LDCU numerical flux was developed for the Euler equations of gas dynamics, which, in the 1-D case, read as (1) with $U = (\rho, \rho u, E)^\top$ and $F = (\rho u, \rho u^2 + p, u(E + p))^\top$. Here, ρ , u , p , and E are the density, velocity, pressure, and total energy, respectively. The system is completed through the following equations of state (EOS) for ideal gases:

$$p = (\gamma - 1) \left[E - \frac{1}{2} \rho u^2 \right],$$

where the parameter γ represents the specific heat ratio.

The LDCU numerical fluxes are computed using the one-sided cell interface values $U_{j+\frac{1}{2}}^\pm$, which are reconstructed from the cell averages $\{\bar{U}_j\}$. In order to suppress oscillations, which may appear near shocks and contact discontinuities, we carry out the reconstruction in local characteristic variables. To this end, we define $\hat{A}_{j+\frac{1}{2}} := A(\hat{U}_{j+\frac{1}{2}})$, and compute the matrices $R_{j+\frac{1}{2}}$ and $R_{j+\frac{1}{2}}^{-1}$ such that $R_{j+\frac{1}{2}}^{-1} \hat{A}_{j+\frac{1}{2}} R_{j+\frac{1}{2}}$ is diagonal. Here, $\hat{U}_{j+\frac{1}{2}}$ is an average of U_j and U_{j+1} , which is obtained by averaging the primitive variables:

$$\hat{\rho}_{j+\frac{1}{2}} = \frac{\bar{\rho}_j + \bar{\rho}_{j+1}}{2}, \quad \hat{u}_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2}, \quad \hat{p}_{j+\frac{1}{2}} = \frac{p_j + p_{j+1}}{2}.$$

We then introduce the local characteristic variables in the neighborhood of $x = x_{j+\frac{1}{2}}$ (see, e.g., [2, 7, 11, 15]):

$$\Gamma_\ell = R_{j+\frac{1}{2}}^{-1} \bar{U}_{j+\ell}, \quad \ell = -1, 0, 1, 2.$$

Using $\Gamma_{-1}, \Gamma_0, \Gamma_1, \Gamma_2$, we compute the slopes using the SBM-type limiters introduced in [9]:

$$(\Gamma_x)_0 = \phi_{\theta, \tau}^{\text{SBM}} \left(\frac{\Gamma_1 - \Gamma_0}{\Gamma_0 - \Gamma_{-1}} \right) \frac{\Gamma_0 - \Gamma_{-1}}{\Delta x} \quad \text{and} \quad (\Gamma_x)_1 = \phi_{\theta, \tau}^{\text{SBM}} \left(\frac{\Gamma_2 - \Gamma_1}{\Gamma_1 - \Gamma_0} \right) \frac{\Gamma_1 - \Gamma_0}{\Delta x}, \quad (4)$$

where the function $\phi_{\theta, \tau}^{\text{SBM}}$, defined by

$$\phi_{\theta, \tau}^{\text{SBM}}(r) := \begin{cases} 0, & \text{if } r < 0, \\ \min \{r\theta, 1 + \tau(r - 1)\}, & \text{if } 0 \leq r \leq 1, \\ r \phi_{\theta, \tau}^{\text{SBM}}\left(\frac{1}{r}\right), & \text{otherwise,} \end{cases} \quad (5)$$

is applied in a componentwise manner. The parameters $\theta \in [1, 2]$ and τ control the properties of the resulting reconstruction. In general, larger values of θ lead to less dissipative, but typically more oscillatory, reconstructions. In all numerical examples in §4, we take $\theta = 2$, which can be safely used when the limiters are applied to the local characteristic variables. The parameter τ determines whether the limiter is dissipative, compressive, or overcompressive: if $\tau \geq 0.5$, then the limiter is dissipative and contact discontinuities are typically smeared; if $0 \leq \tau < 0.5$, then the limiter is compressive and contact waves are resolved sharply, but smooth extrema may be slightly compressed, producing mild kinks; if $\tau < 0$, then the limiter is overcompressive and while contact waves typically remain sharp for long times, smooth profiles may develop artificial $O(1)$ jumps and/or stair-like structures. A particular choice made in [3] was $\theta = 2$, $\tau = 0.5$

(which corresponds to the Minmod2 limiter) and $\theta = 2$, $\tau = -0.25$ in the “smooth” and “rough” areas, respectively.

Once the slopes (4) are available, we compute

$$\Gamma_{\frac{1}{2}}^- = \Gamma_0 + \frac{\Delta x}{2}(\Gamma_x)_0, \quad \Gamma_{\frac{1}{2}}^+ = \Gamma_1 - \frac{\Delta x}{2}(\Gamma_x)_1,$$

and recover the interface values in conservative variables by $U_{j+\frac{1}{2}}^\pm = R_{j+\frac{1}{2}} \Gamma_{\frac{1}{2}}^\pm$.

Remark 1 For the Euler equations of gas dynamics, explicit expressions and implementation details for $R_{j+\frac{1}{2}}$ and $R_{j+\frac{1}{2}}^{-1}$ can be found in [2, Appendix A].

2.2 One-Dimensional Adaptive Scheme

We now turn to the description of the proposed adaptive scheme. The key idea is to continuously vary the parameter $\tau \in [-0.25, 0.5]$ in (5) according to the local smoothness of the computed solution.

To detect the local smoothness, we employ the SI from [10] applied to the density. As in [3], we first define the quantities

$$\mathcal{E}_j := \frac{|\bar{\rho}_{j+1} - 2\bar{\rho}_j + \bar{\rho}_{j-1}|}{|\bar{\rho}_{j+1} - \bar{\rho}_j| + |\bar{\rho}_j - \bar{\rho}_{j-1}| + \varepsilon[|\bar{\rho}_{j+1}| + 2|\bar{\rho}_j| + |\bar{\rho}_{j-1}|]}, \quad (6)$$

and then apply a local averaging to introduce

$$\bar{\mathcal{E}}_j := \frac{1}{6}[\mathcal{E}_{j+1} + 4\mathcal{E}_j + \mathcal{E}_{j-1}].$$

In (6), the parameter ε acts as a “noise” filter that prevents the refinement of small-amplitude wiggles caused by loss of monotonicity. In all of the numerical experiments reported in §4.1, we have taken $\varepsilon = 0.2$ as in [3, 10].

Given $\bar{\mathcal{E}}_j$, we prescribe τ_j as

$$\tau_j = \begin{cases} \frac{1}{8} \left[1 + 3 \tanh(2000(C - \bar{\mathcal{E}}_j)) \right], & \text{if } \bar{\mathcal{E}}_j < C, \\ \frac{1}{8} \left[1 + 3 \tanh(300(C - \bar{\mathcal{E}}_j)) \right], & \text{otherwise,} \end{cases}$$

where $C > 0$ is a tunable parameter. The above formula yields $\tau_j \approx 0.5$ in “smooth” regions, where $\bar{\mathcal{E}}_j \ll C$, $\tau_j \approx -0.25$ in “rough” regions, where $\bar{\mathcal{E}}_j \gg C$, while providing a gradual transition in a zone between “smooth” and “rough” areas. We then use $\tau = \tau_j$ for computing the SBM-type limiters in cell I_j .

Remark 2 The constant C is a positive parameter to be selected for each problem at hand. The robustness of the shock detection strategy depends on how sensitive the algorithm is to the choice of C . In our computations, C is tuned on a coarse mesh, and

the same value is then used on finer meshes. This tuning procedure is demonstrated in Example 2 in §4.1.

Remark 3 The adaptive strategy in [3] employs a discontinuous switching between -0.25 and 0.5 , so that in [3] we have used

$$\tau_j = \begin{cases} -0.25, & \text{if } \bar{\mathcal{E}}_j > C, \\ 0.5, & \text{otherwise.} \end{cases}$$

3 Two-Dimensional Scheme Adaptation Algorithm

In this section, we extend the 1-D adaptive strategy developed in §2 to the 2-D hyperbolic systems of conservation laws (2). We first briefly review the 2-D LDCU scheme from [4] (§3.1) and then introduce the new 2-D adaptive scheme (§3.2).

3.1 2-D Low-Dissipation Central-Upwind (LDCU) Scheme

We introduce a uniform mesh consisting of the cells $I_{j,k} := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$, centered at (x_j, y_k) , where $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $y_k = (y_{k-\frac{1}{2}} + y_{k+\frac{1}{2}})/2$, $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$, and $y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}} \equiv \Delta y$. We denote by $\bar{U}_{j,k}$ the cell averages of U over $I_{j,k}$,

$$\bar{U}_{j,k} \approx \frac{1}{\Delta x \Delta y} \iint_{I_{j,k}} U(x, y, t) \, dy \, dx,$$

and assume that, at a given time level $t \geq 0$, the values $\{\bar{U}_{j,k}\}$ are available.

Following [4], the 2-D LDCU semi-discrete scheme evolves the cell averages by numerically solving the following system of ODEs:

$$\frac{d\bar{U}_{j,k}}{dt} = -\frac{\mathcal{F}_{j+\frac{1}{2},k} - \mathcal{F}_{j-\frac{1}{2},k}}{\Delta x} - \frac{\mathcal{G}_{j,k+\frac{1}{2}} - \mathcal{G}_{j,k-\frac{1}{2}}}{\Delta y}, \quad (7)$$

where $\mathcal{F}_{j+\frac{1}{2},k} = \mathcal{F}(U_{j+\frac{1}{2},k}^-, U_{j+\frac{1}{2},k}^+)$ and $\mathcal{G}_{j,k+\frac{1}{2}} = \mathcal{G}(U_{j,k+\frac{1}{2}}^-, U_{j,k+\frac{1}{2}}^+)$ are the LDCU numerical fluxes from [4] developed for the 2-D Euler equations of gas dynamics, which read as (2) with $U = (\rho, \rho u, \rho v, E)^\top$, $F = (\rho u, \rho u^2 + p, \rho uv, u(E + p))^\top$, and $G = (\rho v, \rho uv, \rho v^2 + p, v(E + p))^\top$. Here, u and v are the x - and y -directional velocities, respectively, and the pressure is given by the ideal-gas EOS

$$p = (\gamma - 1) \left[E - \frac{\rho}{2} (u^2 + v^2) \right].$$

The LDCU numerical fluxes are computed using the one-sided point values $U_{j+\frac{1}{2},k}^\pm$ and $U_{j,k+\frac{1}{2}}^\pm$, which can be reconstructed in local characteristic variables via the local characteristic decomposition; see [3, Appendix A] for details.

3.2 Two-Dimensional Adaptive Scheme

We now turn to the description of the proposed 2-D adaptive scheme. As in the 1-D case, the key idea is to smoothly vary the parameter $\tau \in [-0.25, 0.5]$ in the SBM-type limiters (5) according to the local smoothness of the computed solution.

To detect the local smoothness, we employ the 2-D version of the SI from [10] applied to the density. To this end, we first compute $\mathcal{E}_{j,k} := \sqrt{\mathcal{E}_{j,k}^1 / \mathcal{E}_{j,k}^2}$, where

$$\begin{aligned}\mathcal{E}_{j,k}^1 &:= (\bar{\rho}_{j+1,k} - 2\bar{\rho}_{j,k} + \bar{\rho}_{j-1,k})^2 + (\bar{\rho}_{j,k+1} - 2\bar{\rho}_{j,k} + \bar{\rho}_{j,k-1})^2, \\ \mathcal{E}_{j,k}^2 &:= \left(|\bar{\rho}_{j+1,k} - \bar{\rho}_{j,k}| + |\bar{\rho}_{j,k} - \bar{\rho}_{j-1,k}| + \varepsilon[|\bar{\rho}_{j+1,k}| + 2|\bar{\rho}_{j,k}| + |\bar{\rho}_{j-1,k}|] \right)^2 \\ &\quad + \left(|\bar{\rho}_{j,k+1} - \bar{\rho}_{j,k}| + |\bar{\rho}_{j,k} - \bar{\rho}_{j,k-1}| + \varepsilon[|\bar{\rho}_{j,k+1}| + 2|\bar{\rho}_{j,k}| + |\bar{\rho}_{j,k-1}|] \right)^2,\end{aligned}$$

with ε playing the role of a noise filter; see §2.2 for further discussion. We then apply a local averaging and introduce

$$\begin{aligned}\bar{\mathcal{E}}_{j,k} &:= \frac{1}{36} \left[\mathcal{E}_{j-1,k-1} + \mathcal{E}_{j-1,k+1} + \mathcal{E}_{j+1,k-1} + \mathcal{E}_{j+1,k+1} \right. \\ &\quad \left. + 4(\mathcal{E}_{j-1,k} + \mathcal{E}_{j,k-1} + \mathcal{E}_{j,k+1} + \mathcal{E}_{j+1,k}) + 16\mathcal{E}_{j,k} \right].\end{aligned}$$

Finally, we select $\tau_{j,k}$ by the same smooth mapping as in the 1-D case,

$$\tau_{j,k} = \begin{cases} \frac{1}{8} \left[1 + 3 \tanh(2000(C - \bar{\mathcal{E}}_{j,k})) \right], & \text{if } \bar{\mathcal{E}}_{j,k} < C, \\ \frac{1}{8} \left[1 + 3 \tanh(300(C - \bar{\mathcal{E}}_{j,k})) \right], & \text{otherwise,} \end{cases}$$

where $C > 0$ is a tunable threshold parameter, and use $\tau = \tau_{j,k}$ in cell $I_{j,k}$ to evaluate the SBM-type limiters there.

4 Numerical Examples

In this section, we test the developed adaptive schemes on a series of numerical examples and compare their performance with that of the adaptive LDCU schemes from [3]. To this end, we apply the proposed adaptive schemes to the 1-D and 2-D Euler equations of gas dynamics. For the sake of brevity, the adaptive LDCU schemes from [3] and

the proposed new adaptive schemes will be referred to as the OLD and NEW schemes, respectively.

In all of the numerical experiments, the semi-discrete ODE systems (3) and (7) are integrated in time by the three-stage third-order strong stability preserving (SSP) Runge-Kutta method (see, e.g., [5, 6]) with the CFL number set to 0.4. We take $\gamma = 1.4$ in Examples 1–7 and $\gamma = 5/3$ in Example 8.

4.1 One-Dimensional Examples

Example 1—Shock-Density Wave Interaction Problem. In the first example taken from [18], we consider the shock-density wave interaction problem with the following initial data,

$$(\rho, u, p)|_{(x,0)} = \begin{cases} \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3}\right), & x < -4, \\ (1 + 0.2 \sin(5x), 0, 1), & x > -4, \end{cases}$$

which are prescribed in the computational domain $[-5, 15]$ subject to the free boundary conditions.

We compute the numerical solutions by the OLD (with the adaption constant $C = 0.01$) and NEW (with the adaption constant $C = 0.005$) schemes on a uniform mesh with $\Delta x = 1/40$ until the final time $t = 5$, and plot the obtained numerical results in Figure 1 together with the reference solution computed by the LDCU scheme on a much finer mesh with $\Delta x = 1/400$. It can be clearly seen that the NEW scheme produces a slightly more accurate result compared to that obtained by the OLD scheme.

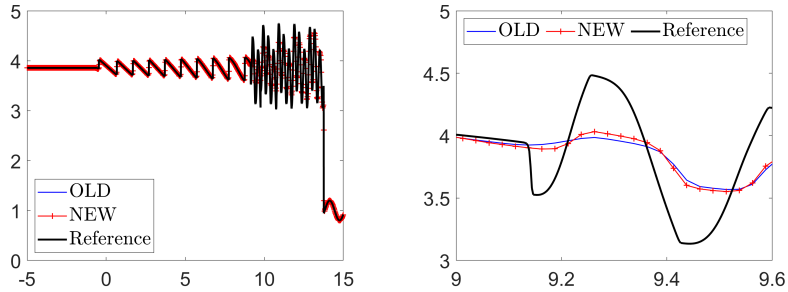


Fig. 1 Example 1: Density ρ computed by the OLD and NEW schemes (left) and zoom at $x \in [9, 9.6]$ (right).

Example 2—Titarev-Toro Problem. In the second example taken from [19] (see also [17, 20]), we consider the shock-entropy wave interaction problem. The initial conditions,

$$(\rho, u, p)\Big|_{(x,0)} = \begin{cases} (1.51695, 0.523346, 1.805), & x < -4.5, \\ (1 + 0.1 \sin(20x), 0, 1), & x > -4.5, \end{cases}$$

correspond to a forward-facing shock wave of Mach number 1.1 interacting with high-frequency density perturbations, that is, as the shock wave moves, the perturbations spread ahead. The free boundary conditions are imposed in the computational domain $[-5, 5]$.

We compute the numerical solution until the final time $t = 5$ by the OLD (with the adaption constant $C = 0.01$) and NEW (with the adaption constant $C = 0.002$) schemes on a uniform mesh with $\Delta x = 1/80$. The obtained numerical results are plotted in Figure 2 along with the reference solution computed by the LDCU scheme on a much finer mesh with $\Delta x = 1/1600$. In Figure 2 (right), we zoom the obtained solutions at the interval $[-2, -1]$, where one can see that this part of the solution is resolved much more accurately by the NEW scheme.

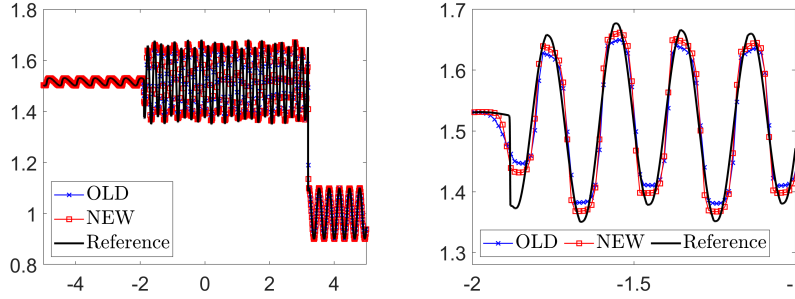


Fig. 2 Example 2: Density ρ computed by the OLD and NEW schemes (left) and zoom at $x \in [-2, -1]$ (right).

As mentioned above, the adaptive coefficient C may be tuned as follows. First, it is adjusted on a coarse grid (this makes the tuning process computationally inexpensive) and then used for high-resolution computation on finer grids. We demonstrate the tuning process in this example. To this end, we first compute the numerical results on a coarse mesh with $\Delta x = 1/20$ with $C = 0.005, 0.002, 0.0005$, and 0.0001 , and plot the corresponding values of $\ln(\bar{\mathcal{E}}_j)$ computed at the final time step in Figure 3. As one can see, the values $C = 0.0005$ and 0.0001 seem to be too small as for them the computed solution will be identified everywhere as “rough” and the overcompressive limiter will be used throughout the entire computational domain. On the other hand, the values $C = 0.005$ and 0.002 seem to be reasonable and the coarse mesh selection in Figure 2 was thus produced with $C = 0.002$. We then compute the corresponding numerical results on a finer mesh with $\Delta x = 1/400$ and plot the obtained numerical results in Figure 4, where one can clearly see that the selection made on a coarse mesh is robust.

Example 3—Blast Wave Problem. In the last 1-D example, we consider a strong shock interaction problem from [21] subject to the following initial conditions:

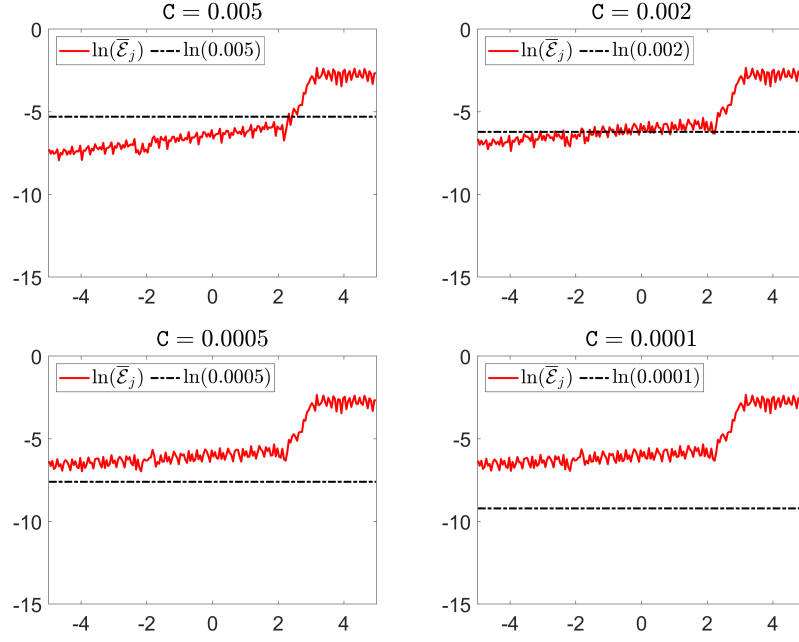


Fig. 3 Example 2: $\ln(\bar{\mathcal{E}}_j)$ computed on a coarse mesh with $\Delta x = 1/20$ and $C = 0.005$ (top left), $C = 0.002$ (top right), $C = 0.0005$ (bottom left), and $C = 0.0001$ (bottom right).

$$(\rho, u, p)\Big|_{(x,0)} = \begin{cases} (1, 0, 1000), & x < 0.1, \\ (1, 0, 0.01), & 0.1 \leq x \leq 0.9, \\ (1, 0, 100), & x > 0.9, \end{cases}$$

which is considered on the interval $[0, 1]$ subject to the solid wall boundary conditions.

We compute the numerical solutions until the final time $t = 0.038$ by the OLD (with the adaption constant $C = 0.01$) and NEW (with the adaption constant $C = 0.005$) schemes on a uniform mesh with $\Delta x = 1/400$. The obtained results, together with the reference solution computed by the LDCU scheme on a much finer mesh with $\Delta x = 1/8000$, are plotted in Figure 5, where one can see that both the OLD and NEW schemes produce similar results. This confirms the robustness of the new scheme adaption strategy: like the OLD scheme, the NEW scheme is capable of achieving superb resolution of the contact wave located at about $x = 0.6$.

4.2 Two-Dimensional Examples

Example 4—2-D Accuracy Test. In the first 2-D example taken from [1, 16], we consider the following smooth initial data:

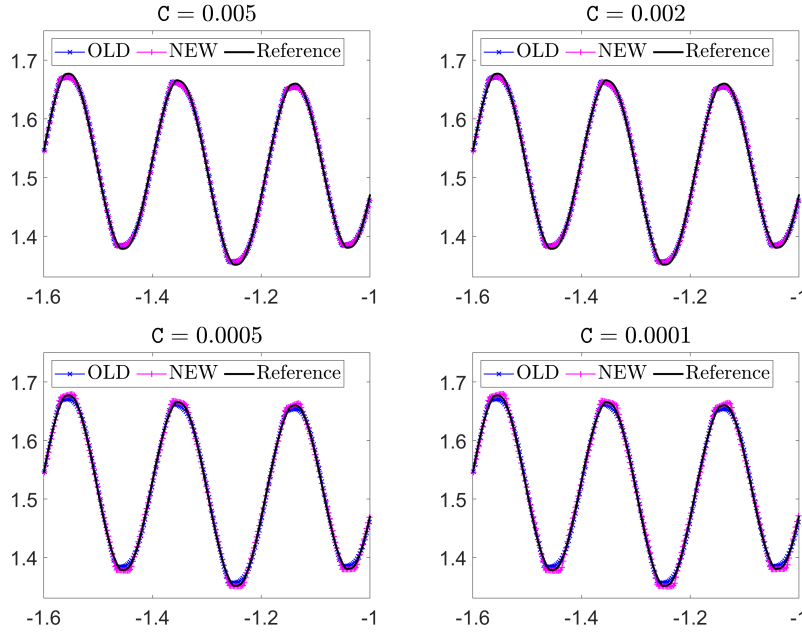


Fig. 4 Example 2: Density ρ (zoom at $x \in [-1.6, -1]$) computed with $\Delta x = 1/400$ and $C = 0.005$ (top left), $C = 0.002$ (top right), $C = 0.0005$ (bottom left), and $C = 0.0001$ (bottom right).

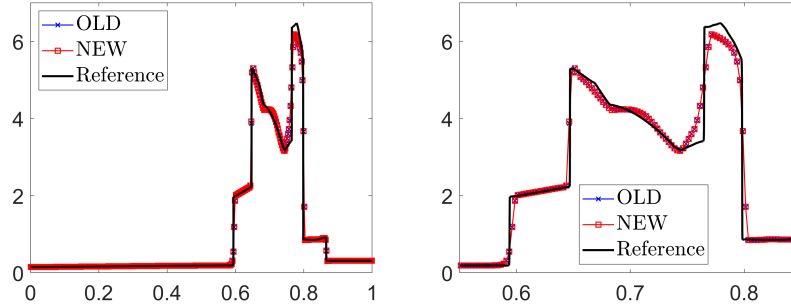


Fig. 5 Example 3: Density ρ computed by the OLD and NEW schemes (left) and zoom at $x \in [0.55, 0.85]$.

$$\rho(x, y, 0) = \left(1 - \frac{(\gamma - 1)\kappa^2}{2\gamma}\right)^{\frac{1}{\gamma-1}}, \quad p(x, y, 0) = \rho^\gamma(x, y, 0),$$

$$u(x, y, 0) = 1 - \kappa y, \quad v(x, y, 0) = 1 + \kappa x, \quad \kappa = \frac{5}{2\pi} e^{0.5(1-x^2-y^2)}$$

subject to the periodic boundary conditions in the computational domain $[-10, 10] \times [-10, 10]$. The exact solution of this initial value problem is given by $U(x, y, t) = U(x - t, y - t, 0)$.

We compute the numerical solution until the final time $t = 0.1$ by the NEW scheme (with the adaptation constant $C = 0.05$) on a sequence of uniform meshes with $\Delta x = \Delta y = 1/10, 1/20, 1/40$, and $1/80$, and then measure the L^1 -errors and the corresponding experimental convergence rates for ρ , u , v , and p . The obtained results are presented in Table 1, where one can see that the expected second order of accuracy has been achieved by the studied NEW scheme.

$\Delta x = \Delta y$	ρ		u		v		p	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/10	3.67e-03	—	6.07e-03	—	6.18e-03	—	4.46e-03	—
1/20	8.34e-04	2.14	1.47e-03	2.04	1.51e-03	2.04	1.00e-03	2.16
1/40	1.74e-04	2.26	3.22e-04	2.19	3.28e-04	2.20	2.07e-04	2.27
1/80	3.65e-05	2.25	6.74e-05	2.25	6.86e-05	2.26	4.28e-05	2.27

Table 1 Example 4: The L^1 -errors of ρ , u , v , and p , and experimental convergence rates for the NEW scheme.

Example 5—2-D Riemann Problem (Configuration 3). In this example, we consider Configuration 3 of the 2-D Riemann problems from [8] (also see [12, 13, 22]) with the initial conditions

$$(\rho, u, v, p)|_{(x,y,0)} = \begin{cases} (1.5, 0, 0, 1.5), & x > 1, y > 1, \\ (0.5323, 1.206, 0, 0.3), & x < 1, y > 1, \\ (0.138, 1.206, 1.206, 0.029), & x < 1, y < 1, \\ (0.5323, 0, 1.206, 0.3), & x > 1, y < 1, \end{cases}$$

prescribed in the computational domain $[0, 1.2] \times [0, 1.2]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t = 1$ by the OLD (with the adaption constant $C = 0.08$) and NEW (with the adaption constant $C = 0.06$) schemes on a uniform mesh with $\Delta x = \Delta y = 3/2500$ and present the obtained results in Figure 6, where one can see that the NEW scheme slightly outperforms the OLD one in capturing a sideband instability of the jet in the zones of strong along-jet velocity shear and the instability near the jet neck.

Example 6—2-D Riemann Problem (Configuration 6). In this 2-D example, we consider Configuration 6 of the 2-D Riemann problems from [8] with the initial conditions

$$(\rho, u, v, p)|_{(x,y,0)} = \begin{cases} (1, 0.75, -0.5, 1), & x > 0.5, y > 0.5, \\ (2, 0.75, 0.5, 1), & x < 0.5, y > 0.5, \\ (1, -0.75, 0.5, 1), & x < 0.5, y < 0.5, \\ (3, -0.75, -0.5, 1), & x > 0.5, y < 0.5, \end{cases}$$

which are prescribed in the computational domain $[0, 1] \times [0, 1]$ subject to the free boundary conditions.

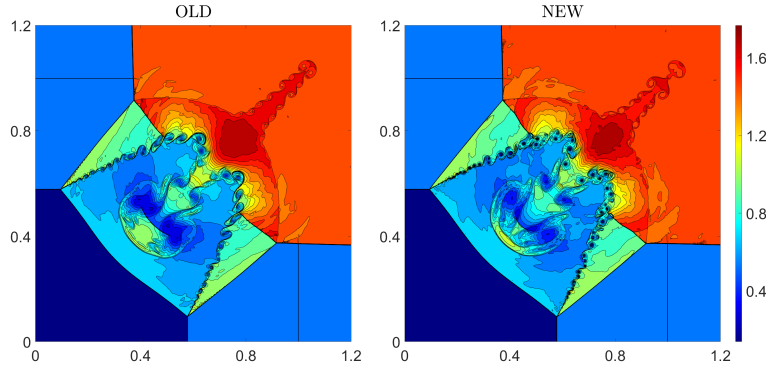


Fig. 6 Example 5: Density ρ computed by the OLD (left) and NEW (right) schemes.

We compute the numerical solution until the final time $t = 1$ by the OLD (with the adaption constant $C = 0.1$) and NEW (with the adaption constant $C = 0.075$) schemes on a uniform mesh with $\Delta x = \Delta y = 1/600$, and plot the obtained results in Figure 7. One can see that the NEW scheme is capable of capturing much more complicated vortex structures compared with the OLD one, which demonstrates higher resolution and lower numerical dissipation of the NEW scheme.

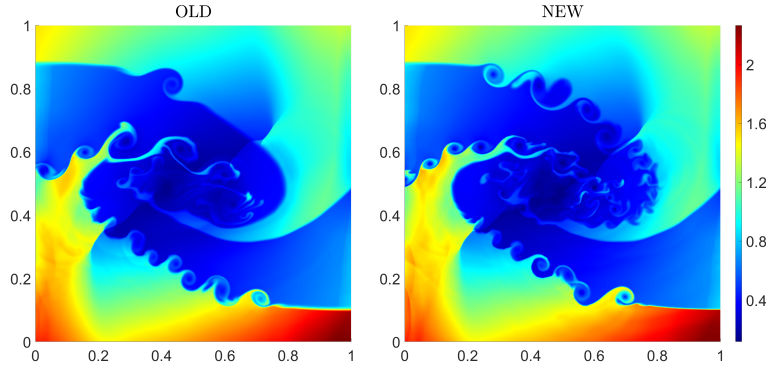


Fig. 7 Example 6: Density ρ computed by the OLD (left) and NEW (right) schemes.

Example 7—2-D Riemann Problem (Configuration 12). In this example, we consider Configuration 12 of the 2-D Riemann problems from [8] with the initial conditions

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (0.5313, 0, 0, 0.4), & x > 0.5, y > 0.5, \\ (1, 0.7276, 0, 1), & x < 0.5, y > 0.5, \\ (0.8, 0, 0, 1), & x < 0.5, y < 0.5, \\ (1, 0, 0.7276, 1), & x > 0.5, y < 0.5, \end{cases}$$

prescribed in the computational domain $[0, 0.6] \times [0, 0.6]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t = 0.5$ by the OLD (with the adaption constant $C = 0.03$) and NEW (with the adaption constant $C = 0.025$) schemes on a uniform mesh with $\Delta x = \Delta y = 1/1000$, and plot the obtained results in Figure 8. As one can see, the NEW scheme resolves more vortices arising along the unstable contact surfaces compared with the OLD scheme.

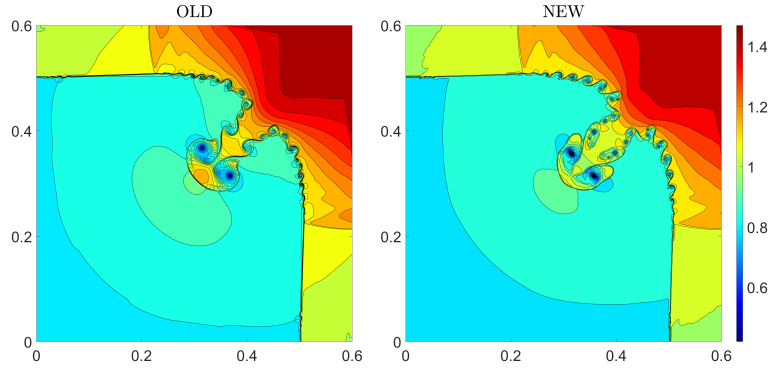


Fig. 8 Example 7: Density ρ computed by the OLD (left) and NEW (right) schemes.

Example 8—RT Instability. In the last example taken from [14], we investigate the RT instability, which is a physical phenomenon occurring when a layer of heavier fluid is placed on top of a layer of lighter fluid. To this end, we first modify the 2-D Euler equations of gas dynamics by adding the gravitational source terms acting in the positive direction of the y -axis into the RHS of the system:

$$\begin{aligned}\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\ (\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= \rho, \\ E_t + [u(E + p)]_x + [v(E + p)]_y &= \rho v,\end{aligned}$$

and then use the following initial conditions:

$$(\rho, u, v, p)\Big|_{(x,y,0)} = \begin{cases} (2, 0, -0.025c \cos(8\pi x), 2y + 1), & y < 0.5, \\ (1, 0, -0.025c \cos(8\pi x), y + 1.5), & \text{otherwise,} \end{cases}$$

where $c := \sqrt{\gamma p / \rho}$ is the speed of sound. The solid wall boundary conditions are imposed at $x = 0$ and $x = 0.25$, and the following Dirichlet boundary conditions are specified at the top and bottom boundaries:

$$(\rho, u, v, p)(x, 1, t) = (1, 0, 0, 2.5), \quad (\rho, u, v, p)(x, 0, t) = (2, 0, 0, 1).$$

We compute the numerical solution until the final time $t = 2.95$ by the OLD (with the adaption constant $C = 0.08$) and NEW (with the adaption constant $C = 0.06$) schemes on a uniform mesh with $\Delta x = \Delta y = 1/1024$ in the computational domain $[0, 0.25] \times [0, 1]$. The obtained numerical results are presented in Figure 9, where one can see that the NEW scheme is less dissipative than the OLD one in capturing the vortices.

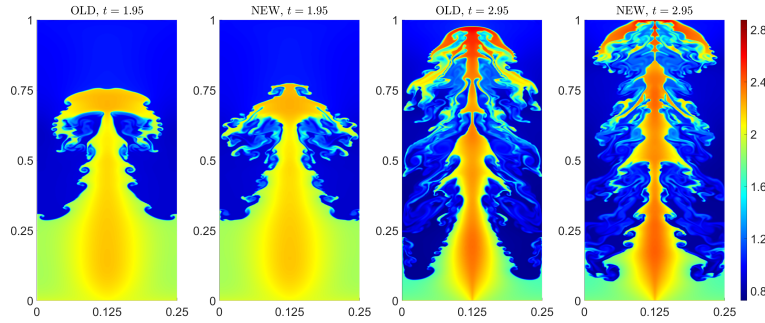


Fig. 9 Example 8: Density ρ computed by the OLD and NEW schemes at $t = 1.95$ and 2.95 .

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