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A Comparative Study of Low-Dissipation Numerical Schemes for Hyperbolic Conservation Laws

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ABSTRACT

This work provides a comparative assessment of several low-dissipation numerical schemes for hyperbolic conservation laws, highlighting their performance relative to the classical Harten-Lax-van Leer (HLL) schemes. The schemes under consideration include the classical Harten-Lax-van Leer-Contact (HLLC), the recently proposed TV flux splitting, the low-dissipation Central-Upwind (LDCU), and the local characteristic decomposition-based Central-Upwind (LCDCU) schemes. These methods are extended to higher orders of accuracy, up to the fifth order, within both finite-volume and finite-difference frameworks. A series of numerical experiments for the one- and two-dimensional Euler equations of gas dynamics are performed to evaluate the accuracy, robustness, and computational efficiency of the studied schemes. The comparison highlights the trade-offs between resolution of contact and shear waves, robustness in the presence of shocks, and computational cost. The investigated low-dissipation schemes show comparable levels of numerical dissipation, with only subtle differences appearing in selected benchmark problems. The results provide practical guidance for selecting efficient low-dissipation solvers for the simulation of complex compressible flows.

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1 | Introduction

This paper focuses on the numerical solution of hyperbolic conservation laws, which in one and two spatial dimensions take the form

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x = \boldsymbol{0}, \tag{1.1}$$

and

$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x + \boldsymbol{G}(\boldsymbol{U})_y = \boldsymbol{0}, \tag{1.2}$$

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where t denotes time, x and y are spatial coordinates, $\boldsymbol{U} \in \mathbb{R}^d$ is the vector of conserved variables, and $\boldsymbol{F}, \boldsymbol{G} : \mathbb{R}^d \to \mathbb{R}^d$ are the flux functions in the x- and y-directions, respectively.

It is well-known that the systems (1.1) and (1.2) may involve complex wave patterns, including shocks, rarefactions, and contact discontinuities, even when the initial data are very smooth, which presents significant challenges for numerical methods. Traditional approaches, particularly first-order methods, often fail to capture these discontinuities accurately and require extremely fine meshes to produce reliable results.

Since the pioneering works [1–3], numerous methods have been developed to solve hyperbolic systems (1.1) and (1.2); for a detailed overview, see, e.g., the monographs and review papers [4–9] and the references therein. Other important high-resolution methods have also been developed for hyperbolic conservation laws. For instance, the Roe approximate Riemann solver from [10] resolves different characteristic waves based on a linearized Riemann problem. Another important class of high-order schemes is based on ENO/WENO reconstructions; see, e.g., [11, 12]. Runge–Kutta discontinuous Galerkin methods provide an alternative high-order framework and have been widely applied to compressible flow problems; see, e.g., [13, 14]. Entropy-stable schemes [15] enforce a discrete entropy inequality, while residual-distribution methods [16] construct multidimensional discretizations by distributing local residuals among the degrees of freedom. These methods represent important developments in this research area and provide a broader background for the low-dissipation schemes studied in this paper. The Harten-Lax-van Leer (HLL) solver from Reference [17] is one of the most widely used approximate Riemann solvers for hyperbolic conservation laws. It replaces the exact Riemann solution, which may involve multiple nonlinear waves, by a simple two-wave model. Specifically, the solution is approximated by a single intermediate state separated from the left and right data by two waves propagating with estimated minimum and maximum signal velocities. The HLL flux is robust and, under suitable signal-speed estimates and a CFL restriction, positivity-preserving, but the main drawback is that intermediate waves such as contact and shear discontinuities are excessively smeared, since they are not explicitly represented in the two-wave model. To overcome this limitation, the Harten-Lax-van Leer-Contact (HLLC) solver was introduced in Reference [18], which extends the HLL framework by restoring the missing contact wave. The HLLC scheme thus consists of three waves, separating the solution into two intermediate states. The presence of the contact wave allows the solver to resolve contact discontinuities and shear layers more sharply, while maintaining the robustness and efficiency of the HLL scheme. Owing to this balance between accuracy and robustness, the HLLC flux has become one of the most popular approximate Riemann solvers for the compressible Euler equations and related hyperbolic systems.

Another important class of schemes is based on flux vector splitting. In the flux splitting methods, one can decompose fluxes into components corresponding to different wave families, such as shock, contact, and expansion waves, which makes them fundamental in computational fluid dynamics. This decomposition enhances the resolution of shock waves and discontinuities, which is crucial for accurate simulations of compressible flows; see, e.g., [19–29]. However, the classical flux splitting schemes (e.g., [19, 20, 27, 28]) often struggle to resolve intermediate characteristic fields, leading to excessive dissipation or numerical artifacts, even though they are effective. To overcome these limitations, more advanced methods, such as the advection upstream splitting method (AUSM), have been developed (see, e.g., [26]), offering better handling of contact waves and improved resolution of wave patterns. AUSM has since garnered considerable attention and undergone refinement, with further developments [23–25]. Additionally, a flux splitting approach similar to AUSM was proposed in Reference [29], and subsequent advancements have been recorded in works such as [21, 22, 30, 31], further enriching the field of computational fluid dynamics. Recently, a new flux splitting method, known as the TV splitting scheme for the one-dimensional (1-D) Euler equations of gas dynamics, was introduced in Reference [32] and later extended to higher dimensions and orders in References [33, 34]. The proposed schemes are characterized by their simplicity, robustness, and accuracy, offering significant improvements over existing flux splitting methods. In particular, they effectively capture contact and shear waves while precisely preserving isolated stationary contacts. Beyond their applications to high-dimensional Euler equations, they have also been extended to other systems, including magnetohydrodynamics and shallow water equations; see, e.g., [35–38].

While flux splitting methods such as TV splitting focus on decomposing fluxes along characteristic fields, an alternative direction is offered by central schemes. These methods avoid explicit Riemann solvers altogether and instead compute fluxes through staggered or modified control volumes [39–44]. Central schemes are easy to implement and broadly applicable, but they tend to suffer from relatively large numerical dissipation. To overcome this drawback, central-upwind (CU) schemes were developed in References [45, 46], in which control volumes are adapted to local wave speeds. CU schemes retain the simplicity of central methods while achieving significantly improved accuracy. Further refinements include anti-diffusion corrections [47], dissipation switches [48], and more accurate wave-speed estimates [49]. Recently,

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two refinements of CU schemes have been proposed to further reduce dissipation. Low-dissipation CU (LDCU) schemes [50, 51] introduce a novel projection step based on subcell resolution, which sharply approximates contact waves while ensuring non-oscillatory behavior of the projected solution; see also, e.g., [52–54]. This significantly decreases dissipation in contact- and shear-dominated regions. In parallel, LCD-based CU (LCDCU) schemes [55] apply local characteristic decomposition to the diffusion term, which also leads to substantially improved resolution by further suppressing excessive numerical dissipation; see also, e.g., [56, 57]. An important issue common to low-dissipation shock-capturing methods is the balance between resolution and nonlinear robustness. Excessive numerical dissipation smears contact discontinuities and shear waves, whereas insufficient or inappropriately distributed dissipation may allow perturbations along grid-aligned strong shocks to grow, producing odd–even decoupling or carbuncle-type instabilities. The relation of these instabilities to the construction of numerical mass fluxes and upwind dissipation has been analyzed in References [58, 59] and has motivated shock-stable modifications of contact-resolving solvers [60]. Consequently, a meaningful comparison should assess contact and shear-wave resolution, robustness in the presence of strong shocks, preservation of physically admissible states, and computational efficiency together.

Against this background, we carry out a systematic comparative study of the first-, second-, third-, and fifth-order HLL, HLLC, TV splitting, LDCU, and LCDCU schemes. The comparison focuses on three closely related aspects: The resolution of contact discontinuities and shear waves, robustness in the presence of strong and multidimensional shocks, and computational efficiency. Since no single dissipation mechanism is expected to be uniformly optimal for all flow regimes, the purpose of this study is not to identify a universally superior method, but to clarify the practical strengths and limitations of the selected schemes across a broad range of benchmark problems. The second-order extension employs a piecewise linear interpolant for reconstructing one-sided point values, while the third- and fifth-order schemes are formulated within the finite-difference (FD) A-WENO framework. This framework effectively generalizes low-order finite-volume (FV) schemes to higher-order FD ones, especially in multidimensional settings, through its simple dimension-by-dimension reconstruction; see, e.g., [61–63].

The rest of this paper is organized as follows. In Section 2, we begin with a brief overview of the first-order HLL scheme for the 1-D Euler equations of gas dynamics and then extend it to second-, third-, and fifth-order accuracy. The corresponding HLLC, TV, LDCU, and LCDCU schemes are subsequently reviewed. In Section 3, we describe the two-dimensional (2-D) extensions of these schemes in a dimension-by-dimension manner. Finally, in Section 4, we present a number of 1-D and 2-D numerical results to compare their performance.

2 | One-Dimensional Schemes

In this section, we consider the 1-D Euler equations of gas dynamics and briefly review the first-, second-, third-, and fifth-order 1-D HLL, HLLC, TV, LDCU, and LCDCU schemes, respectively. The 1-D Euler equations of gas dynamics read as (1.1) with

$$\boldsymbol{U} := (\rho, \rho u, E)^{\top}, \quad \text{and} \quad \boldsymbol{F}(\boldsymbol{U}) = (\rho u, \rho u^2 + p, u(E+p))^{\top}, \tag{2.1}$$

where ρ, u, p, and E are the density, velocity, pressure, and total energy, respectively. The system is completed through the following equation of state (EOS):

$$p = (\gamma - 1)\left[E - \frac{1}{2}\rho u^2\right], \tag{2.2}$$

where the parameter γ represents the specific heat ratio.

2.1 | One-Dimensional HLL Schemes

We first briefly review the 1-D HLL scheme from Reference [17] and show its high-order extensions.

2.1.1 | One-Dimensional First-Order HLL Scheme

Assume the computational domain is covered with uniform cells $C_j := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ with $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$ centered at $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$, $j = 1, \ldots, N$, and the cell average values

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$$\overline{\boldsymbol{U}}_j(t) :\approx \frac{1}{\Delta x}\int_{C_j} \boldsymbol{U}(x,t)\,\mathrm{d}x$$

are available at a certain time level t. The computed cell averages $\overline{\boldsymbol{U}}_j$ of the 1-D system (1.1) are evolved in time by solving the following semi-discrete system of ordinary differential equations (ODEs):

$$\frac{\mathrm{d}\overline{\boldsymbol{U}}_j}{\mathrm{d}t} = -\frac{\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} - \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2}}}{\Delta x}, \tag{2.3}$$

where $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} = \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}\right)$ is the numerical flux, defined by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}(\boldsymbol{U}^-_{j+\frac{1}{2}}, \boldsymbol{U}^+_{j+\frac{1}{2}}) = \begin{cases} \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}), & a^-_{j+\frac{1}{2}} \geq 0, \\ \dfrac{a^+_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}}\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}) + a^+_{j+\frac{1}{2}}a^-_{j+\frac{1}{2}}(\boldsymbol{U}^+_{j+\frac{1}{2}} - \boldsymbol{U}^-_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}, & a^-_{j+\frac{1}{2}} \leq 0 \leq a^+_{j+\frac{1}{2}}, \\ \boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2}}), & a^+_{j+\frac{1}{2}} \leq 0, \end{cases} \tag{2.4}$$

where $\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}$ are the left/right-sided point values of $\boldsymbol{U}$ at the cell interfaces $x_{j+\frac{1}{2}}$. For the first order scheme, we take $\boldsymbol{U}^+_{j+\frac{1}{2}} = \boldsymbol{U}_{j+1}$ and $\boldsymbol{U}^-_{j+\frac{1}{2}} = \boldsymbol{U}_j$. The one-sided local speeds of propagation $a^{\pm}_{j+\frac{1}{2}}$ can be estimated by

$$a^+_{j+\frac{1}{2}} = \max\left\{u^+_{j+\frac{1}{2}} + c^+_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}} + c^-_{j+\frac{1}{2}}\right\}, \quad a^-_{j+\frac{1}{2}} = \min\left\{u^+_{j+\frac{1}{2}} - c^+_{j+\frac{1}{2}}, u^-_{j+\frac{1}{2}} - c^-_{j+\frac{1}{2}}\right\}, \tag{2.5}$$

where

$$u^{\pm}_{j+\frac{1}{2}} = \frac{(\rho u)^{\pm}_{j+\frac{1}{2}}}{\rho^{\pm}_{j+\frac{1}{2}}}, \quad p^{\pm}_{j+\frac{1}{2}} = (\gamma-1)\left[E^{\pm}_{j+\frac{1}{2}} - \frac{1}{2}\rho^{\pm}_{j+\frac{1}{2}}\left(u^{\pm}_{j+\frac{1}{2}}\right)^2\right], \quad c^{\pm}_{j+\frac{1}{2}} = \sqrt{\frac{\gamma p^{\pm}_{j+\frac{1}{2}}}{\rho^{\pm}_{j+\frac{1}{2}}}}.$$

2.1.2 | One-Dimensional Second-Order HLL Scheme

We now extend the first-order HLL scheme introduced in Subsection 2.1 to the second order of accuracy. The resulting scheme (2.3) to (2.5) achieves second-order accuracy provided that the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}$, used to compute the numerical flux $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$, are second-order accurate. To this end, we approximate $\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}$ using the generalized minmod reconstruction [44, 64, 65] with $\theta = 1.3$.

2.1.3 | One-Dimensional Third-Order HLL Scheme

In this section, we extend the HLL scheme to third-order accuracy in the framework of the FD A-WENO scheme introduced in Reference [62] (see also [63]), which has been proven to be a powerful tool for generalizing low-order FV schemes to higher-order FD ones.

Following [62], the point values $\boldsymbol{U}_j$ are evolved in time by solving the following system of ODEs:

$$\frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t} = -\frac{\boldsymbol{H}_{j+\frac{1}{2}} - \boldsymbol{H}_{j-\frac{1}{2}}}{\Delta x}, \tag{2.6}$$

where $\boldsymbol{H}_{j+\frac{1}{2}}$ is the (third-order accurate) numerical flux defined by

$$\boldsymbol{H}_{j+\frac{1}{2}} = \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} - \frac{1}{24}(\Delta x)^2(\boldsymbol{F}_{xx})_{j+\frac{1}{2}}. \tag{2.7}$$

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Here, $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ is the FV numerical flux as in Equation (2.4) and $(\boldsymbol{F}_{xx})_{j+\frac{1}{2}}$ is the higher-order correction term used to increase the order of the numerical flux. The correction term $(\boldsymbol{F}_{xx})_{j+\frac{1}{2}}$ can be approximated with the help of the FV numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$:

$$(\boldsymbol{F}_{xx})_{j+\frac{1}{2}} = \frac{1}{(\Delta x)^2}\left[\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2}} - 2\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} + \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{3}{2}}\right], \tag{2.8}$$

which has been proved to be more efficient than the old version

$$(\boldsymbol{F}_{xx})_{j+\frac{1}{2}} = \frac{1}{2(\Delta x)^2}\left[\boldsymbol{F}_{j-1} - \boldsymbol{F}_{j} - \boldsymbol{F}_{j+1} + \boldsymbol{F}_{j+2}\right], \tag{2.9}$$

where $\boldsymbol{F}_j = \boldsymbol{F}(\boldsymbol{U}_j)$, while affecting neither the accuracy nor the quality of resolution. The resulting scheme is third-order once the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}$ employed to compute the numerical flux $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ are third-order accurate. This can be done by implementing a certain nonlinear limiting procedure like the third-order WENO-type interpolation (see, e.g., [66–69]) applied to the local characteristic variables; see [33], Appendix A for a detailed explanation.

2.1.4 | One-Dimensional Fifth-Order HLL Scheme

According to [62], to achieve fifth-order accuracy, the point values $\boldsymbol{U}_j$ are evolved in time by solving the system (2.6) with the (fifth-order accurate) numerical flux

$$\boldsymbol{H}_{j+\frac{1}{2}} = \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} - \frac{1}{24}(\Delta x)^2(\boldsymbol{F}_{xx})_{j+\frac{1}{2}} + \frac{7}{5760}(\Delta x)^4(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2}}, \tag{2.10}$$

where $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ is the FV numerical flux as in Equation (2.4), $(\boldsymbol{F}_{xx})_{j+\frac{1}{2}}$ and $(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2}}$ are the higher-order correction terms computed by the fourth- and second-order accurate FDs, respectively; see, e.g., [70, 71]. Here, we have used the following higher-order correction terms from Reference [71]:

$$\begin{aligned}
(\boldsymbol{F}_{xx})_{j+\frac{1}{2}} &= \frac{1}{12(\Delta x)^2}\left[-\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{3}{2}} + 16\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2}} - 30\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} + 16\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{3}{2}} - \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{5}{2}}\right],\\
(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2}} &= \frac{1}{(\Delta x)^4}\left[\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{3}{2}} - 4\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2}} + 6\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}} - 4\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{3}{2}} + \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{5}{2}}\right].
\end{aligned} \tag{2.11}$$

To ensure the resulting scheme is fifth order, the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2}}$ employed to compute the numerical flux $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ need to be at least fifth-order accurate. This can be done by using a certain nonlinear limiting procedure like the fifth-order WENO-Z interpolation from References [62, 63, 72] applied to the local characteristic variables; see [33], Appendix B for details.

2.2 | One-Dimensional HLLC Schemes

According to [18], the 1-D first-, second-, third-, and fifth-order HLLC schemes can be obtained similarly to the corresponding HLL schemes by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ in Equation (2.4) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}) = \begin{cases}
\boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2}}), & a^{-}_{j+\frac{1}{2}} \geq 0,\\
\boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2}}) + a^{-}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{*,-}_{j+\frac{1}{2}} - \boldsymbol{U}^{-}_{j+\frac{1}{2}}\right), & a^{-}_{j+\frac{1}{2}} \leq 0 \leq a^{*}_{j+\frac{1}{2}},\\
\boldsymbol{F}(\boldsymbol{U}^{+}_{j+\frac{1}{2}}) + a^{+}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{*,+}_{j+\frac{1}{2}} - \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right), & a^{*}_{j+\frac{1}{2}} \leq 0 \leq a^{+}_{j+\frac{1}{2}},\\
\boldsymbol{F}(\boldsymbol{U}^{+}_{j+\frac{1}{2}}), & a^{+}_{j+\frac{1}{2}} \leq 0,
\end{cases} \tag{2.12}$$

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where

$$\boldsymbol{U}^{*,\pm}_{j+\frac{1}{2}} = \rho^{\pm}_{j+\frac{1}{2}} \frac{a^{\pm}_{j+\frac{1}{2}} - u^{\pm}_{j+\frac{1}{2}}}{a^{\pm}_{j+\frac{1}{2}} - a^{*}_{j+\frac{1}{2}}} \begin{pmatrix} 1 \\ a^{*}_{j+\frac{1}{2}} \\ \frac{E^{\pm}_{j+\frac{1}{2}}}{\rho^{\pm}_{j+\frac{1}{2}}} + \left(a^{*}_{j+\frac{1}{2}} - u^{\pm}_{j+\frac{1}{2}}\right)\left(a^{*}_{j+\frac{1}{2}} + \frac{p^{\pm}_{j+\frac{1}{2}}}{\rho^{\pm}_{j+\frac{1}{2}}\left(a^{\pm}_{j+\frac{1}{2}} - u^{\pm}_{j+\frac{1}{2}}\right)}\right) \end{pmatrix}, \tag{2.13}$$

and

$$a^{*}_{j+\frac{1}{2}} = \frac{p^{+}_{j+\frac{1}{2}} - p^{-}_{j+\frac{1}{2}} + \rho^{-}_{j+\frac{1}{2}} u^{-}_{j+\frac{1}{2}} (a^{-}_{j+\frac{1}{2}} - u^{-}_{j+\frac{1}{2}}) - \rho^{+}_{j+\frac{1}{2}} u^{+}_{j+\frac{1}{2}} (a^{+}_{j+\frac{1}{2}} - u^{+}_{j+\frac{1}{2}})}{\rho^{-}_{j+\frac{1}{2}} (a^{-}_{j+\frac{1}{2}} - u^{-}_{j+\frac{1}{2}}) - \rho^{+}_{j+\frac{1}{2}} (a^{+}_{j+\frac{1}{2}} - u^{+}_{j+\frac{1}{2}})}. \tag{2.14}$$

2.3 | One-Dimensional TV Splitting Schemes

According to [32, 33], the 1-D first-, second-, third-, and fifth-order TV schemes can be obtained by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ in Equation (2.4) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right) = \boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right) + \boldsymbol{\mathcal{F}}^{P}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right). \tag{2.15}$$

Here, $\boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right)$ is the advection flux given by

$$\boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right) = \begin{cases} u^{*}_{j+\frac{1}{2}} \begin{pmatrix} \rho^{-}_{j+\frac{1}{2}} \\ (\rho u)^{-}_{j+\frac{1}{2}} \\ \frac{1}{2}\rho^{-}_{j+\frac{1}{2}} (u^{-}_{j+\frac{1}{2}})^2 \end{pmatrix}, & \text{if } u^{*}_{j+\frac{1}{2}} \geq 0, \\ u^{*}_{j+\frac{1}{2}} \begin{pmatrix} \rho^{+}_{j+\frac{1}{2}} \\ (\rho u)^{+}_{j+\frac{1}{2}} \\ \frac{1}{2}\rho^{+}_{j+\frac{1}{2}} (u^{+}_{j+\frac{1}{2}})^2 \end{pmatrix}, & \text{otherwise,} \end{cases} \tag{2.16}$$

and $\boldsymbol{\mathcal{F}}^{P}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right)$ is the pressure flux given by

$$\boldsymbol{\mathcal{F}}^{P}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}, \boldsymbol{U}^{+}_{j+\frac{1}{2}}\right) = \left(0,\ p^{*}_{j+\frac{1}{2}},\ \frac{\gamma u^{*}_{j+\frac{1}{2}} p^{*}_{j+\frac{1}{2}}}{\gamma - 1}\right)^{\mathsf{T}}, \tag{2.17}$$

where

$$\begin{aligned} u^{*}_{j+\frac{1}{2}} &= \frac{C^{+}_{j+\frac{1}{2}} u^{+}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}} u^{-}_{j+\frac{1}{2}}}{C^{+}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}}} - \frac{2}{C^{+}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}}}(p^{+}_{j+\frac{1}{2}} - p^{-}_{j+\frac{1}{2}}), \\ p^{*}_{j+\frac{1}{2}} &= \frac{C^{+}_{j+\frac{1}{2}} p^{-}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}} p^{+}_{j+\frac{1}{2}}}{C^{+}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}}} + \frac{C^{+}_{j+\frac{1}{2}} C^{-}_{j+\frac{1}{2}}}{2(C^{+}_{j+\frac{1}{2}} - C^{-}_{j+\frac{1}{2}})}(u^{+}_{j+\frac{1}{2}} - u^{-}_{j+\frac{1}{2}}), \\ C^{\pm}_{j+\frac{1}{2}} &= \rho^{\pm}_{j+\frac{1}{2}} \left(u^{\pm}_{j+\frac{1}{2}} \pm \sqrt{\left(u^{\pm}_{j+\frac{1}{2}}\right)^2 + 4\left(c^{\pm}_{j+\frac{1}{2}}\right)^2}\right). \end{aligned} \tag{2.18}$$

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2.4 | One-Dimensional LDCU Schemes

According to [50–52], the 1-D first-, second-, third-, and fifth-order LDCU schemes can be obtained by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ in Equation (2.4) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}},\boldsymbol{U}^{+}_{j+\frac{1}{2}}\right)=\frac{a^{+}_{j+\frac{1}{2}}\boldsymbol{F}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}}\right)-a^{-}_{j+\frac{1}{2}}\boldsymbol{F}\left(\boldsymbol{U}^{+}_{j+\frac{1}{2}}\right)}{a^{+}_{j+\frac{1}{2}}-a^{-}_{j+\frac{1}{2}}}+\frac{a^{+}_{j+\frac{1}{2}}a^{-}_{j+\frac{1}{2}}}{a^{+}_{j+\frac{1}{2}}-a^{-}_{j+\frac{1}{2}}}\left(\boldsymbol{U}^{+}_{j+\frac{1}{2}}-\boldsymbol{U}^{-}_{j+\frac{1}{2}}\right)+\boldsymbol{q}_{j+\frac{1}{2}},\tag{2.19}$$

and the anti-diffusion term $\boldsymbol{q}_{j+\frac{1}{2}}$ is given by [50].

Finally, the one-sided local speeds of propagation $a^{\pm}_{j+\frac{1}{2}}$ are slightly different from Equation (2.5), and are defined by

$$a^{+}_{j+\frac{1}{2}}=\max\left\{u^{+}_{j+\frac{1}{2}}+c^{+}_{j+\frac{1}{2}},u^{-}_{j+\frac{1}{2}}+c^{-}_{j+\frac{1}{2}},0\right\},\quad a^{-}_{j+\frac{1}{2}}=\min\left\{u^{+}_{j+\frac{1}{2}}-c^{+}_{j+\frac{1}{2}},u^{-}_{j+\frac{1}{2}}-c^{-}_{j+\frac{1}{2}},0\right\}.\tag{2.20}$$

2.5 | One-Dimensional LCDCU Schemes

According to [55, 70], the 1-D first-, second-, third-, and fifth-order LCDCU schemes can be obtained by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}$ in Equation (2.4) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2}},\boldsymbol{U}^{+}_{j+\frac{1}{2}}\right)=R_{j+\frac{1}{2}}P^{\mathrm{LCD}}_{j+\frac{1}{2}}R^{-1}_{j+\frac{1}{2}}\boldsymbol{F}^{-}_{j+\frac{1}{2}}+R_{j+\frac{1}{2}}M^{\mathrm{LCD}}_{j+\frac{1}{2}}R^{-1}_{j+\frac{1}{2}}\boldsymbol{F}^{+}_{j+\frac{1}{2}}+R_{j+\frac{1}{2}}Q^{\mathrm{LCD}}_{j+\frac{1}{2}}R^{-1}_{j+\frac{1}{2}}\left(\boldsymbol{U}^{+}_{j+\frac{1}{2}}-\boldsymbol{U}^{-}_{j+\frac{1}{2}}\right).$$

The diagonal matrices $P^{\mathrm{LCD}}_{j+\frac{1}{2}}$, $M^{\mathrm{LCD}}_{j+\frac{1}{2}}$, and $Q^{\mathrm{LCD}}_{j+\frac{1}{2}}$ are given by

$$P^{\mathrm{LCD}}_{j+\frac{1}{2}}=\mathrm{diag}\left((P^{\mathrm{LCD}}_{1})_{j+\frac{1}{2}},\ldots,(P^{\mathrm{LCD}}_{d})_{j+\frac{1}{2}}\right),\quad M^{\mathrm{LCD}}_{j+\frac{1}{2}}=\mathrm{diag}\left((M^{\mathrm{LCD}}_{1})_{j+\frac{1}{2}},\ldots,(M^{\mathrm{LCD}}_{d})_{j+\frac{1}{2}}\right),$$
$$Q^{\mathrm{LCD}}_{j+\frac{1}{2}}=\mathrm{diag}\left((Q^{\mathrm{LCD}}_{1})_{j+\frac{1}{2}},\ldots,(Q^{\mathrm{LCD}}_{d})_{j+\frac{1}{2}}\right)$$

with

$$\left((P^{\mathrm{LCD}}_{i})_{j+\frac{1}{2}},(M^{\mathrm{LCD}}_{i})_{j+\frac{1}{2}},(Q^{\mathrm{LCD}}_{i})_{j+\frac{1}{2}}\right)$$
$$=\begin{cases}\frac{1}{(\lambda^{+}_{i})_{j+\frac{1}{2}}-(\lambda^{-}_{i})_{j+\frac{1}{2}}}\left((\lambda^{+}_{i})_{j+\frac{1}{2}},-(\lambda^{-}_{i})_{j+\frac{1}{2}},(\lambda^{+}_{i})_{j+\frac{1}{2}}(\lambda^{-}_{i})_{j+\frac{1}{2}}\right) & \text{if } (\lambda^{+}_{i})_{j+\frac{1}{2}}-(\lambda^{-}_{i})_{j+\frac{1}{2}}>\varepsilon_0,\\ \left(\frac{1}{2},\frac{1}{2},0\right) & \text{otherwise,}\end{cases}$$

where

$$(\lambda^{+}_{i})_{j+\frac{1}{2}}=\max\left\{\lambda_i\left(A(\boldsymbol{U}^{-}_{j+\frac{1}{2}})\right),\ \lambda_i\left(A(\boldsymbol{U}^{+}_{j+\frac{1}{2}})\right),\ 0\right\},\quad (\lambda^{-}_{i})_{j+\frac{1}{2}}=\min\left\{\lambda_i\left(A(\boldsymbol{U}^{-}_{j+\frac{1}{2}})\right),\ \lambda_i\left(A(\boldsymbol{U}^{+}_{j+\frac{1}{2}})\right),\ 0\right\},$$

for $i=1,\ldots,d$. Here, $A(\boldsymbol{U})=\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}(\boldsymbol{U})$ is the Jacobian, $\lambda_1(A(\boldsymbol{U}))\le\ldots\lambda_d(A(\boldsymbol{U}))$ are its eigenvalues, and the matrices $R_{j+\frac{1}{2}}$ and $R^{-1}_{j+\frac{1}{2}}$ are such that $R^{-1}_{j+\frac{1}{2}}\widehat{A}_{j+\frac{1}{2}}R_{j+\frac{1}{2}}$ is diagonal, where $\widehat{A}_{j+\frac{1}{2}}=A(\widehat{\boldsymbol{U}}_{j+\frac{1}{2}})$ and $\widehat{\boldsymbol{U}}_{j+\frac{1}{2}}$ is either a simple average $(\overline{\boldsymbol{U}}_j+\overline{\boldsymbol{U}}_{j+1})/2$ or another type of average of the $\overline{\boldsymbol{U}}_j$ and $\overline{\boldsymbol{U}}_{j+1}$ states. Finally, ε_0 is a very small desingularization constant, which is taken $\varepsilon_0=10^{-16}$ in all of the numerical examples reported in Section 4.

3 | Two-Dimensional Schemes

We now present a 2-D overview of the HLL, HLLC, TV, LDCU, and LCDCU schemes for the 2-D Euler equations of gas dynamics, which read as

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$$\boldsymbol{U}_t + \boldsymbol{F}(\boldsymbol{U})_x + \boldsymbol{G}(\boldsymbol{U})_y = \boldsymbol{0}, \tag{3.1}$$

with $\boldsymbol{U} := (\rho, \rho u, \rho v, E)^\top$, $\boldsymbol{F}(\boldsymbol{U}) = (\rho u, \rho u^2 + p, \rho uv, u(E+p))^\top$, and $\boldsymbol{G}(\boldsymbol{U}) = (\rho v, \rho uv, \rho v^2 + p, v(E+p))^\top$. Here, v is the y-velocity and the rest of the notation is the same as in the 1-D case (2.1) and (2.2). The system is completed through the following EOS:

$$p = (\gamma - 1)\left[E - \frac{\rho}{2}(u^2 + v^2)\right]. \tag{3.2}$$

3.1 | Two-Dimensional HLL Schemes

As in the 1-D case, we first briefly review the first-order 2-D HLL scheme from Reference [17] and show its high-order extensions.

3.1.1 | Two-Dimensional First-Order HLL Scheme

Supposing that the computational domain is covered with uniform cells $C_{j,\,k} := [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{k-\frac{1}{2}}, y_{k+\frac{1}{2}}]$ centered at $(x_j, y_k) = \left((x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2, (y_{k+\frac{1}{2}} + y_{k-\frac{1}{2}})/2\right)$ with $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \equiv \Delta x$ and $y_{k+\frac{1}{2}} - y_{k-\frac{1}{2}} \equiv \Delta y$ for all j, k, we assume that the cell averages

$$\overline{\boldsymbol{U}}_{j,k}(t) :\approx \frac{1}{\Delta x \Delta y} \int_{C_{j,k}} \boldsymbol{U}(x, y, t)\, \mathrm{d}x\mathrm{d}y$$

are available at a certain time level t. The cell averages $\overline{\boldsymbol{U}}_{j,k}$ are then evolved in time by numerically solving the following system of ODEs:

$$\frac{\mathrm{d}\overline{\boldsymbol{U}}_{j,k}}{\mathrm{d}t} = -\frac{\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k} - \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2},k}}{\Delta x} - \frac{\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}} - \boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k-\frac{1}{2}}}{\Delta y}. \tag{3.3}$$

Here, $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^-_{j+\frac{1}{2},k}, \boldsymbol{U}^+_{j+\frac{1}{2},k}\right)$ and $\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^-_{j,k+\frac{1}{2}}, \boldsymbol{U}^+_{j,k+\frac{1}{2}}\right)$ are the numerical fluxes, defined by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}(\boldsymbol{U}^-_{j+\frac{1}{2},k}, \boldsymbol{U}^+_{j+\frac{1}{2},k}) = \begin{cases} \boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2},k}), & a^-_{j+\frac{1}{2},k} \geq 0, \\ \dfrac{a^+_{j+\frac{1}{2},k}\boldsymbol{F}(\boldsymbol{U}^-_{j+\frac{1}{2},k}) - a^-_{j+\frac{1}{2},k}\boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2},k}) + a^+_{j+\frac{1}{2},k}a^-_{j+\frac{1}{2},k}(\boldsymbol{U}^+_{j+\frac{1}{2},k} - \boldsymbol{U}^-_{j+\frac{1}{2},k})}{a^+_{j+\frac{1}{2},k} - a^-_{j+\frac{1}{2},k}}, & a^-_{j+\frac{1}{2},k} \leq 0 \leq a^+_{j+\frac{1}{2},k}, \\ \boldsymbol{F}(\boldsymbol{U}^+_{j+\frac{1}{2},k}), & a^+_{j+\frac{1}{2},k} \leq 0, \end{cases} \tag{3.4}$$

$$\boldsymbol{\mathcal{G}}^{\mathrm{FV}}(\boldsymbol{U}^-_{j,k+\frac{1}{2}}, \boldsymbol{U}^+_{j,k+\frac{1}{2}}) = \begin{cases} \boldsymbol{G}(\boldsymbol{U}^-_{j,k+\frac{1}{2}}), & b^-_{j,k+\frac{1}{2}} \geq 0, \\ \dfrac{b^+_{j,k+\frac{1}{2}}\boldsymbol{G}(\boldsymbol{U}^-_{j,k+\frac{1}{2}}) - b^-_{j,k+\frac{1}{2}}\boldsymbol{G}(\boldsymbol{U}^+_{j,k+\frac{1}{2}}) + b^+_{j,k+\frac{1}{2}}b^-_{j,k+\frac{1}{2}}(\boldsymbol{U}^+_{j,k+\frac{1}{2}} - \boldsymbol{U}^-_{j,k+\frac{1}{2}})}{b^+_{j,k+\frac{1}{2}} - b^-_{j,k+\frac{1}{2}}}, & b^-_{j,k+\frac{1}{2}} \leq 0 \leq b^+_{j,k+\frac{1}{2}}, \\ \boldsymbol{G}(\boldsymbol{U}^+_{j,k+\frac{1}{2}}), & b^+_{j,k+\frac{1}{2}} \leq 0, \end{cases} \tag{3.5}$$

where $\boldsymbol{U}^\pm_{j+\frac{1}{2},k}$ and $\boldsymbol{U}^\pm_{j,k+\frac{1}{2}}$ are the left/right-sided point values of $\boldsymbol{U}$ at the cell interfaces $(x_{j+\frac{1}{2}}, y_k)$ and $(x_j, y_{k+\frac{1}{2}})$, respectively. In the first-order scheme, we take $\boldsymbol{U}^+_{j+\frac{1}{2},k} = \boldsymbol{U}_{j+1,k}$, $\boldsymbol{U}^+_{j,k+\frac{1}{2}} = \boldsymbol{U}_{j,k+1}$, and $\boldsymbol{U}^-_{j+\frac{1}{2},k} = \boldsymbol{U}^-_{j,k+\frac{1}{2}} = \boldsymbol{U}_{j,k}$. Here, the one-sided local speeds of propagation $a^\pm_{j+\frac{1}{2},k}$ and $b^\pm_{j,k+\frac{1}{2}}$ can be estimated in a "dimension-by-dimension" manner to the x- and y-directions respectively as in the 1-D case. Here, we omit the details for the sake of brevity.

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3.1.2 | Two-Dimensional Second-Order HLL Scheme

As in the 1-D case, the resulting scheme (3.3) to (3.5) is second-order accurate once the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2},k}$ and $\boldsymbol{U}^{\pm}_{j,k+\frac{1}{2}}$ employed to compute the numerical fluxes (3.4) and (3.5) are second order. To this end, we approximate $\boldsymbol{U}^{\pm}_{j+\frac{1}{2},k}$ and $\boldsymbol{U}^{\pm}_{j,k+\frac{1}{2}}$ by the generalized minmod reconstruction [44, 64, 65] with $\theta = 1.3$ in the x- and y- directions, respectively.

3.1.3 | Two-Dimensional Third-Order HLL Scheme

Following [62], the point values $\boldsymbol{U}_{j,k}$ are evolved in time by solving the following system of ODEs:

$$\frac{\mathrm{d}\boldsymbol{U}_{j,k}}{\mathrm{d}t} = -\frac{\boldsymbol{H}_{j+\frac{1}{2},k} - \boldsymbol{H}_{j-\frac{1}{2},k}}{\Delta x} - \frac{\boldsymbol{H}_{j,k+\frac{1}{2}} - \boldsymbol{H}_{j,k-\frac{1}{2}}}{\Delta y}, \tag{3.6}$$

where the numerical fluxes $\boldsymbol{H}_{j+\frac{1}{2},k}$ and $\boldsymbol{H}_{j,k+\frac{1}{2}}$ are defined by

$$\boldsymbol{H}_{j+\frac{1}{2},k} = \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k} - \frac{1}{24}(\Delta x)^2(\boldsymbol{F}_{xx})_{j+\frac{1}{2},k}, \quad \boldsymbol{H}_{j,k+\frac{1}{2}} = \boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}} - \frac{1}{24}(\Delta y)^2(\boldsymbol{G}_{yy})_{j,k+\frac{1}{2}}.$$

Here, $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\boldsymbol{G}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ are the FV numerical fluxes as in Equations (3.4) and (3.5), $(\boldsymbol{F}_{xx})_{j+\frac{1}{2},k}$ and $(\boldsymbol{G}_{yy})_{j,k+\frac{1}{2}}$ are the higher-order correction terms computed by the numerical fluxes

$$\begin{aligned}
(\boldsymbol{F}_{xx})_{j+\frac{1}{2},k} &= \frac{1}{(\Delta x)^2}\left[\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j-\frac{1}{2},k} - 2\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k} + \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{3}{2},k}\right],\\
(\boldsymbol{G}_{yy})_{j,k+\frac{1}{2}} &= \frac{1}{(\Delta y)^2}\left[\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k-\frac{1}{2}} - 2\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}} + \boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{3}{2}}\right].
\end{aligned} \tag{3.7}$$

To ensure the resulting scheme is third-order accurate, the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2},k}$ and $\boldsymbol{U}^{\pm}_{j,k+\frac{1}{2}}$ are also computed using third-order WENO-type interpolation applied to the local characteristic variables. Note that this can be done in a "dimension-by-dimension" manner as in the 1-D case; we therefore omit the details for the sake of brevity.

3.1.4 | Two-Dimensional Fifth-Order HLL Scheme

According to [62], the point values $\boldsymbol{U}_{j,k}$ are evolved in time by solving the system of ODEs (3.6) with the following numerical fluxes $\boldsymbol{H}_{j+\frac{1}{2},k}$ and $\boldsymbol{H}_{j,k+\frac{1}{2}}$:

$$\begin{aligned}
\boldsymbol{H}_{j+\frac{1}{2},k} &= \boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k} - \frac{1}{24}(\Delta x)^2(\boldsymbol{F}_{xx})_{j+\frac{1}{2},k} + \frac{7}{5760}(\Delta x)^4(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2},k},\\
\boldsymbol{H}_{j,k+\frac{1}{2}} &= \boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}} - \frac{1}{24}(\Delta y)^2(\boldsymbol{G}_{yy})_{j,k+\frac{1}{2}} + \frac{7}{5760}(\Delta y)^4(\boldsymbol{G}_{yyyy})_{j,k+\frac{1}{2}}.
\end{aligned}$$

Here, $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\boldsymbol{G}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ are the FV numerical fluxes as in Equations (3.4) and (3.5), $(\boldsymbol{F}_{xx})_{j+\frac{1}{2},k}$, $(\boldsymbol{F}_{xxxx})_{j+\frac{1}{2},k}$, $(\boldsymbol{G}_{yy})_{j,k+\frac{1}{2}}$, $(\boldsymbol{G}_{yyyy})_{j,k+\frac{1}{2}}$ are approximations of the second- and fourth-order spatial derivatives of $\boldsymbol{F}$ at $(x, y) = (x_{j+\frac{1}{2}}, y_k)$ and $\boldsymbol{G}$ at $(x, y) = (x_j, y_{k+\frac{1}{2}})$, respectively. The higher-order correction terms are obtained from the one-dimensional formulas in Equation (2.11) by applying the same finite-difference approximations dimension by dimension.

To achieve fifth-order accuracy, the one-sided point values $\boldsymbol{U}^{\pm}_{j+\frac{1}{2},k}$ and $\boldsymbol{U}^{\pm}_{j,k+\frac{1}{2}}$ employed to compute the numerical flux $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ need to be at least fifth-order accurate. This can also be done in a "dimension-by-dimension" manner as in the 1-D case; we therefore omit the details for the sake of brevity.

3.2 | Two-Dimensional HLLC Schemes

According to [18], the two-dimensional first-, second-, third-, and fifth-order HLLC schemes are obtained by replacing the numerical fluxes in Equations (3.4) and (3.5) with the corresponding HLLC fluxes. The x-direction numerical flux $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ is given by

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$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k})=\begin{cases}\boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2},k}), & a^{-}_{j+\frac{1}{2},k}\geq 0,\\ \boldsymbol{F}(\boldsymbol{U}^{-}_{j+\frac{1}{2},k})+a^{-}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{*,-}_{j+\frac{1}{2},k}-\boldsymbol{U}^{-}_{j+\frac{1}{2},k}\right), & a^{-}_{j+\frac{1}{2},k}\leq 0\leq a^{*}_{j+\frac{1}{2},k},\\ \boldsymbol{F}(\boldsymbol{U}^{+}_{j+\frac{1}{2},k})+a^{+}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{*,+}_{j+\frac{1}{2},k}-\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right), & a^{*}_{j+\frac{1}{2},k}\leq 0\leq a^{+}_{j+\frac{1}{2},k},\\ \boldsymbol{F}(\boldsymbol{U}^{+}_{j+\frac{1}{2},k}), & a^{+}_{j+\frac{1}{2},k}\leq 0,\end{cases}$$

where

$$\boldsymbol{U}^{*,\pm}_{j+\frac{1}{2},k}=\rho^{\pm}_{j+\frac{1}{2},k}\frac{a^{\pm}_{j+\frac{1}{2},k}-u^{\pm}_{j+\frac{1}{2},k}}{a^{\pm}_{j+\frac{1}{2},k}-a^{*}_{j+\frac{1}{2},k}}\begin{pmatrix}1\\ a^{*}_{j+\frac{1}{2},k}\\ v^{\pm}_{j+\frac{1}{2},k}\\ \frac{E^{\pm}_{j+\frac{1}{2},k}}{\rho^{\pm}_{j+\frac{1}{2},k}}+\left(a^{*}_{j+\frac{1}{2},k}-u^{\pm}_{j+\frac{1}{2},k}\right)\left(a^{*}_{j+\frac{1}{2},k}+\frac{p^{\pm}_{j+\frac{1}{2},k}}{\rho^{\pm}_{j+\frac{1}{2},k}\left(a^{\pm}_{j+\frac{1}{2},k}-u^{\pm}_{j+\frac{1}{2},k}\right)}\right)\end{pmatrix},\tag{3.8}$$

and

$$a^{*}_{j+\frac{1}{2},k}=\frac{p^{+}_{j+\frac{1}{2},k}-p^{-}_{j+\frac{1}{2},k}+\rho^{-}_{j+\frac{1}{2},k}u^{-}_{j+\frac{1}{2},k}(a^{-}_{j+\frac{1}{2},k}-u^{-}_{j+\frac{1}{2},k})-\rho^{+}_{j+\frac{1}{2},k}u^{+}_{j+\frac{1}{2},k}(a^{+}_{j+\frac{1}{2},k}-u^{+}_{j+\frac{1}{2},k})}{\rho^{-}_{j+\frac{1}{2},k}(a^{-}_{j+\frac{1}{2},k}-u^{-}_{j+\frac{1}{2},k})-\rho^{+}_{j+\frac{1}{2},k}(a^{+}_{j+\frac{1}{2},k}-u^{+}_{j+\frac{1}{2},k})}.$$

The corresponding y-direction numerical flux $\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ is obtained analogously by interchanging the roles of the x- and y-directions. We therefore omit the details here.

3.3 | Two-Dimensional TV Splitting Schemes

According to [33, 34], the 2-D first-, second-, third-, and fifth-order TV schemes can be obtained by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ in Equations (3.4) and (3.5) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right)=\boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right)+\boldsymbol{\mathcal{F}}^{P}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right),$$
$$\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}},\boldsymbol{U}^{+}_{j,k+\frac{1}{2}}\right)=\boldsymbol{\mathcal{G}}^{A}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}},\boldsymbol{U}^{+}_{j,k+\frac{1}{2}}\right)+\boldsymbol{\mathcal{G}}^{P}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}},\boldsymbol{U}^{+}_{j,k+\frac{1}{2}}\right).$$

Here, $\boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right)$ is the x-direction advection flux given by

$$\boldsymbol{\mathcal{F}}^{A}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right)=\begin{cases}u^{*}_{j+\frac{1}{2},k}\begin{pmatrix}\rho^{-}_{j+\frac{1}{2},k}\\ (\rho u)^{-}_{j+\frac{1}{2},k}\\ (\rho v)^{-}_{j+\frac{1}{2},k}\\ \frac{1}{2}\rho^{-}_{j+\frac{1}{2}}\left[(u^{-}_{j+\frac{1}{2},k})^{2}+(v^{-}_{j+\frac{1}{2},k})^{2}\right]\end{pmatrix}, & \text{if } u^{*}_{j+\frac{1}{2},k}\geq 0,\\ u^{*}_{j+\frac{1}{2},k}\begin{pmatrix}\rho^{+}_{j+\frac{1}{2},k}\\ (\rho u)^{+}_{j+\frac{1}{2},k}\\ (\rho v)^{+}_{j+\frac{1}{2},k}\\ \frac{1}{2}\rho^{+}_{j+\frac{1}{2}}\left[(u^{+}_{j+\frac{1}{2},k})^{2}+(v^{+}_{j+\frac{1}{2},k})^{2}\right]\end{pmatrix}, & \text{otherwise,}\end{cases}$$

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and $\mathcal{F}^P_{j+\frac{1}{2},k}\left(\boldsymbol{U}^-_{j+\frac{1}{2},k}, \boldsymbol{U}^+_{j+\frac{1}{2},k}\right)$ is the x-direction pressure flux given by

$$\mathcal{F}^P_{j+\frac{1}{2},k}\left(\boldsymbol{U}^-_{j+\frac{1}{2},k}, \boldsymbol{U}^+_{j+\frac{1}{2},k}\right) = \left(0,\ p^*_{j+\frac{1}{2},k},\ 0,\ \frac{\gamma u^*_{j+\frac{1}{2},k}\ p^*_{j+\frac{1}{2},k}}{\gamma-1}\right)^\top,$$

where

$$\begin{aligned}
u^*_{j+\frac{1}{2},k} &= \frac{C^+_{j+\frac{1}{2},k}u^+_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k}u^-_{j+\frac{1}{2},k}}{C^+_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k}} - \frac{2}{C^+_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k}}\left(p^+_{j+\frac{1}{2},k} - p^-_{j+\frac{1}{2},k}\right),\\
p^*_{j+\frac{1}{2},k} &= \frac{C^+_{j+\frac{1}{2},k}p^-_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k}p^+_{j+\frac{1}{2},k}}{C^+_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k}} + \frac{C^+_{j+\frac{1}{2},k}C^-_{j+\frac{1}{2},k}}{2(C^+_{j+\frac{1}{2},k} - C^-_{j+\frac{1}{2},k})}\left(u^+_{j+\frac{1}{2},k} - u^-_{j+\frac{1}{2},k}\right),\\
C^\pm_{j+\frac{1}{2},k} &= \rho^\pm_{j+\frac{1}{2},k}\left(u^\pm_{j+\frac{1}{2},k} \pm \sqrt{\left(u^\pm_{j+\frac{1}{2},k}\right)^2 + 4\left(c^\pm_{j+\frac{1}{2},k}\right)^2}\right).
\end{aligned}$$

Similarly, the detailed formulas for the y-direction advection and pressure fluxes $\mathcal{G}^A_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^-_{j,k+\frac{1}{2}}, \boldsymbol{U}^+_{j,k+\frac{1}{2}}\right)$ and $\mathcal{G}^P_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^-_{j,k+\frac{1}{2}}, \boldsymbol{U}^+_{j,k+\frac{1}{2}}\right)$ are obtained by interchanging the roles of the x- and y-directions. We therefore omit them here.

3.4 | Two-Dimensional LDCU Schemes

According to [50–52], the 2-D first-, second-, third-, and fifth-order LDCU schemes can be obtained by replacing the numerical fluxes $\mathcal{F}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\mathcal{G}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ in Equations (3.4) and (3.5) by

$$\begin{aligned}
\mathcal{F}^{\mathrm{FV}}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^-_{j+\frac{1}{2},k}, \boldsymbol{U}^+_{j+\frac{1}{2},k}\right) &= \frac{a^+_{j+\frac{1}{2},k}\boldsymbol{F}\left(\boldsymbol{U}^-_{j+\frac{1}{2},k}\right) - a^-_{j+\frac{1}{2},k}\boldsymbol{F}\left(\boldsymbol{U}^+_{j+\frac{1}{2},k}\right)}{a^+_{j+\frac{1}{2},k} - a^-_{j+\frac{1}{2},k}}\\
&\quad + \frac{a^+_{j+\frac{1}{2},k}a^-_{j+\frac{1}{2},k}}{a^+_{j+\frac{1}{2},k} - a^-_{j+\frac{1}{2},k}}\left(\boldsymbol{U}^+_{j+\frac{1}{2},k} - \boldsymbol{U}^-_{j+\frac{1}{2},k}\right) + \boldsymbol{q}^x_{j+\frac{1}{2},k},\\
\mathcal{G}^{\mathrm{FV}}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^-_{j,k+\frac{1}{2}}, \boldsymbol{U}^+_{j,k+\frac{1}{2}}\right) &= \frac{b^+_{j,k+\frac{1}{2}}\boldsymbol{G}\left(\boldsymbol{U}^-_{j,k+\frac{1}{2}}\right) - b^-_{j,k+\frac{1}{2}}\boldsymbol{G}\left(\boldsymbol{U}^+_{j,k+\frac{1}{2}}\right)}{b^+_{j,k+\frac{1}{2}} - b^-_{j,k+\frac{1}{2}}}\\
&\quad + \frac{b^+_{j,k+\frac{1}{2}}b^-_{j,k+\frac{1}{2}}}{b^+_{j,k+\frac{1}{2}} - b^-_{j,k+\frac{1}{2}}}\left(\boldsymbol{U}^+_{j,k+\frac{1}{2}} - \boldsymbol{U}^-_{j,k+\frac{1}{2}}\right) + \boldsymbol{q}^y_{j,k+\frac{1}{2}},
\end{aligned}$$

where $\boldsymbol{q}^x_{j+\frac{1}{2},k}$ and $\boldsymbol{q}^y_{j,k+\frac{1}{2}}$ are the x- and y-direction anti-diffusion terms, which are given by [50]. The one-sided local speeds of propagation $a^\pm_{j+\frac{1}{2},k}$ and $b^\pm_{j,k+\frac{1}{2}}$ are defined by

$$\begin{aligned}
a^+_{j+\frac{1}{2},k} &= \max\left\{u^+_{j+\frac{1}{2},k} + c^+_{j+\frac{1}{2},k}, u^-_{j+\frac{1}{2},k} + c^-_{j+\frac{1}{2},k}, 0\right\},\quad a^-_{j+\frac{1}{2},k} = \min\left\{u^+_{j+\frac{1}{2},k} - c^+_{j+\frac{1}{2},k}, u^-_{j+\frac{1}{2},k} - c^-_{j+\frac{1}{2},k}, 0\right\},\\
b^+_{j,k+\frac{1}{2}} &= \max\left\{v^+_{j,k+\frac{1}{2}} + c^+_{j,k+\frac{1}{2}}, v^-_{j,k+\frac{1}{2}} + c^-_{j,k+\frac{1}{2}}, 0\right\},\quad b^-_{j,k+\frac{1}{2}} = \min\left\{v^+_{j,k+\frac{1}{2}} - c^+_{j,k+\frac{1}{2}}, v^-_{j,k+\frac{1}{2}} - c^-_{j,k+\frac{1}{2}}, 0\right\}.
\end{aligned} \tag{3.9}$$

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3.5 | Two-Dimensional LCDCU Schemes

According to [55, 70], the 2-D first-, second-, third-, and fifth-order LCDCU schemes can be obtained by replacing the numerical fluxes $\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}$ and $\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}$ in Equations (3.4) and (3.5) by

$$\boldsymbol{\mathcal{F}}^{\mathrm{FV}}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{-}_{j+\frac{1}{2},k},\boldsymbol{U}^{+}_{j+\frac{1}{2},k}\right)=R_{j+\frac{1}{2}}P^{\mathrm{LCD}}_{j+\frac{1}{2},k}R^{-1}_{j+\frac{1}{2},k}\boldsymbol{F}^{-}_{j+\frac{1}{2},k}+R_{j+\frac{1}{2},k}M^{\mathrm{LCD}}_{j+\frac{1}{2},k}R^{-1}_{j+\frac{1}{2},k}\boldsymbol{F}^{+}_{j+\frac{1}{2},k}+R_{j+\frac{1}{2},k}Q^{\mathrm{LCD}}_{j+\frac{1}{2},k}R^{-1}_{j+\frac{1}{2},k}\left(\boldsymbol{U}^{+}_{j+\frac{1}{2},k}-\boldsymbol{U}^{-}_{j+\frac{1}{2},k}\right),$$

$$\boldsymbol{\mathcal{G}}^{\mathrm{FV}}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}},\boldsymbol{U}^{+}_{j,k+\frac{1}{2}}\right)=R_{j,k+\frac{1}{2}}P^{\mathrm{LCD}}_{j,k+\frac{1}{2}}R^{-1}_{j,k+\frac{1}{2}}\boldsymbol{G}^{-}_{j,k+\frac{1}{2}}+R_{j,k+\frac{1}{2}}M^{\mathrm{LCD}}_{j,k+\frac{1}{2}}R^{-1}_{j,k+\frac{1}{2}}\boldsymbol{G}^{+}_{j,k+\frac{1}{2}}+R_{j,k+\frac{1}{2}}Q^{\mathrm{LCD}}_{j,k+\frac{1}{2}}R^{-1}_{j,k+\frac{1}{2}}\left(\boldsymbol{U}^{+}_{j,k+\frac{1}{2}}-\boldsymbol{U}^{-}_{j,k+\frac{1}{2}}\right).$$

The matrices $R_{j+\frac{1}{2},k}$, $R^{-1}_{j+\frac{1}{2},k}$ and $R_{j,k+\frac{1}{2}}$, $R^{-1}_{j,k+\frac{1}{2}}$ are the matrices such that $R^{-1}_{j+\frac{1}{2},k}\widehat{A}_{j+\frac{1}{2},k}R_{j+\frac{1}{2},k}$ and $R^{-1}_{j,k+\frac{1}{2}}\widehat{B}_{j,k+\frac{1}{2}}R_{j,k+\frac{1}{2}}$ are diagonal. Here, $\widehat{A}_{j+\frac{1}{2},k}=A(\widehat{\boldsymbol{U}}_{j+\frac{1}{2},k})$, $\widehat{B}_{j,k+\frac{1}{2}}=B(\widehat{\boldsymbol{U}}_{j,k+\frac{1}{2}})$ with $A(\boldsymbol{U})=\frac{\partial \boldsymbol{F}(\boldsymbol{U})}{\partial \boldsymbol{U}}$, $B(\boldsymbol{U})=\frac{\partial \boldsymbol{G}(\boldsymbol{U})}{\partial \boldsymbol{U}}$, and $\widehat{\boldsymbol{U}}_{j+\frac{1}{2},k}$, $\widehat{\boldsymbol{U}}_{j,k+\frac{1}{2}}$ are either simple averages $(\overline{\boldsymbol{U}}_{j,k}+\overline{\boldsymbol{U}}_{j+1,k})/2$, $(\overline{\boldsymbol{U}}_{j,k}+\overline{\boldsymbol{U}}_{j,k+1})/2$ or another type of averages of $\overline{\boldsymbol{U}}_{j,k}$, $\overline{\boldsymbol{U}}_{j+1,k}$ and $\overline{\boldsymbol{U}}_{j,k}$, $\overline{\boldsymbol{U}}_{j,k+1}$ states, respectively. The diagonal matrices $P^{\mathrm{LCD}}_{j+\frac{1}{2},k}$, $M^{\mathrm{LCD}}_{j+\frac{1}{2},k}$, $Q^{\mathrm{LCD}}_{j+\frac{1}{2},k}$, $P^{\mathrm{LCD}}_{j,k+\frac{1}{2}}$, $M^{\mathrm{LCD}}_{j,k+\frac{1}{2}}$, and $Q^{\mathrm{LCD}}_{j,k+\frac{1}{2}}$ are defined by

$$P^{\mathrm{LCD}}_{j+\frac{1}{2},k}=\mathrm{diag}\left(\left(P^{\mathrm{LCD}}_{1}\right)_{j+\frac{1}{2},k},\ldots,\left(P^{\mathrm{LCD}}_{d}\right)_{j+\frac{1}{2},k}\right),\qquad P^{\mathrm{LCD}}_{j,k+\frac{1}{2}}=\mathrm{diag}\left(\left(P^{\mathrm{LCD}}_{1}\right)_{j,k+\frac{1}{2}},\ldots,\left(P^{\mathrm{LCD}}_{d}\right)_{j,k+\frac{1}{2}}\right),$$

$$M^{\mathrm{LCD}}_{j+\frac{1}{2},k}=\mathrm{diag}\left(\left(M^{\mathrm{LCD}}_{1}\right)_{j+\frac{1}{2},k},\ldots,\left(M^{\mathrm{LCD}}_{d}\right)_{j+\frac{1}{2},k}\right),\qquad M^{\mathrm{LCD}}_{j,k+\frac{1}{2}}=\mathrm{diag}\left(\left(M^{\mathrm{LCD}}_{1}\right)_{j,k+\frac{1}{2}},\ldots,\left(M^{\mathrm{LCD}}_{d}\right)_{j,k+\frac{1}{2}}\right),$$

$$Q^{\mathrm{LCD}}_{j+\frac{1}{2},k}=\mathrm{diag}\left(\left(Q^{\mathrm{LCD}}_{1}\right)_{j+\frac{1}{2},k},\ldots,\left(Q^{\mathrm{LCD}}_{d}\right)_{j+\frac{1}{2},k}\right),\qquad Q^{\mathrm{LCD}}_{j,k+\frac{1}{2}}=\mathrm{diag}\left(\left(Q^{\mathrm{LCD}}_{1}\right)_{j,k+\frac{1}{2}},\ldots,\left(Q^{\mathrm{LCD}}_{d}\right)_{j,k+\frac{1}{2}}\right),$$

where

$$\left(\left(P^{\mathrm{LCD}}_{i}\right)_{j+\frac{1}{2},k},\left(M^{\mathrm{LCD}}_{i}\right)_{j+\frac{1}{2},k},\left(Q^{\mathrm{LCD}}_{i}\right)_{j+\frac{1}{2},k}\right)=\begin{cases}\frac{1}{\Delta(\lambda_i)_{j+\frac{1}{2},k}}\left((\lambda^{+}_{i})_{j+\frac{1}{2},k},-(\lambda^{-}_{i})_{j+\frac{1}{2},k},(\lambda^{+}_{i})_{j+\frac{1}{2},k}(\lambda^{-}_{i})_{j+\frac{1}{2},k}\right) & \text{if } \Delta(\lambda_i)_{j+\frac{1}{2},k}>\varepsilon_0,\\ \left(\frac{1}{2},\frac{1}{2},0\right) & \text{otherwise,}\end{cases}$$

$$\left(\left(P^{\mathrm{LCD}}_{i}\right)_{j,k+\frac{1}{2}},\left(M^{\mathrm{LCD}}_{i}\right)_{j,k+\frac{1}{2}},\left(Q^{\mathrm{LCD}}_{i}\right)_{j,k+\frac{1}{2}}\right)=\begin{cases}\frac{1}{\Delta(\mu_i)_{j,k+\frac{1}{2}}}\left((\mu^{+}_{i})_{j,k+\frac{1}{2}},-(\mu^{-}_{i})_{j,k+\frac{1}{2}},(\mu^{+}_{i})_{j,k+\frac{1}{2}}(\mu^{-}_{i})_{j,k+\frac{1}{2}}\right) & \text{if } \Delta(\mu_i)_{j,k+\frac{1}{2}}>\varepsilon_0,\\ \left(\frac{1}{2},\frac{1}{2},0\right) & \text{otherwise.}\end{cases}$$

Here, $\Delta(\lambda_i)_{j+\frac{1}{2},k}:=(\lambda^{+}_{i})_{j+\frac{1}{2},k}-(\lambda^{-}_{i})_{j+\frac{1}{2},k}$, $\Delta(\mu_i)_{j,k+\frac{1}{2}}:=(\mu^{+}_{i})_{j,k+\frac{1}{2}}-(\mu^{-}_{i})_{j,k+\frac{1}{2}}$, and

$$(\lambda^{+}_{i})_{j+\frac{1}{2},k}=\max\left\{\lambda_i\left(A(\boldsymbol{U}^{-}_{j+\frac{1}{2},k})\right),\ \lambda_i\left(A(\boldsymbol{U}^{+}_{j+\frac{1}{2},k})\right),\ 0\right\},$$

$$(\lambda^{-}_{i})_{j+\frac{1}{2},k}=\min\left\{\lambda_i\left(A(\boldsymbol{U}^{-}_{j+\frac{1}{2},k})\right),\ \lambda_i\left(A(\boldsymbol{U}^{+}_{j+\frac{1}{2},k})\right),\ 0\right\},$$

$$(\mu^{+}_{i})_{j,k+\frac{1}{2}}=\max\left\{\mu_i\left(B(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}})\right),\ \mu_i\left(B(\boldsymbol{U}^{+}_{j,k+\frac{1}{2}})\right),\ 0\right\},$$

$$(\mu^{-}_{i})_{j,k+\frac{1}{2}}=\min\left\{\mu_i\left(B(\boldsymbol{U}^{-}_{j,k+\frac{1}{2}})\right),\ \mu_i\left(B(\boldsymbol{U}^{+}_{j,k+\frac{1}{2}})\right),\ 0\right\},$$

where λ_i and μ_i are the eigenvalues of the Jacobians $A(\boldsymbol{U})$ and $B(\boldsymbol{U})$: $\lambda_1(A)\le\ldots\le\lambda_d(A)$ and $\mu_1(B)\le\ldots\le\mu_d(B)$, respectively.

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4 | Numerical Examples

In this section, we test the studied first-, second-, third-, and fifth-order schemes on several numerical examples and compare their performance. For the sake of brevity, these schemes will be referred to as the first-, second-, third-, and fifth-order schemes, respectively.

We numerically integrate the ODE systems (2.3), (2.6), (3.3), and (3.6) by the three-stage third-order strong stability preserving Runge-Kutta (SSP RK3) method (see, e.g., [73, 74]) and use the CFL number 0.45.

4.1 | One-Dimensional Examples

We begin with the 1-D Euler equations of gas dynamics (1.1), (2.1) and (2.2). In Examples 1–8, we take the specific heat ratio $\gamma = 1.4$.

4.1.1 | Example 1—1-D Accuracy Test

In the first example taken from Reference [33], we consider the system (1.1), (2.1) and (2.2) subject to the following periodic initial conditions,

$$\rho(x,0) = 1 + \frac{1}{5}\sin(2\pi x), \quad u(x,0) \equiv 1, \quad p(x,0) \equiv 1.$$

The exact solution of this initial value problem is given by

$$\rho(x,t) = 1 + \frac{1}{5}\sin[2\pi(x-t)], \quad u(x,t) \equiv 1, \quad p(x,t) \equiv 1.$$

We first compute the numerical solution on the computational domain $[-1,1]$ until the final time $t = 0.1$ using the first-, second-, third-, and fifth-order schemes on a sequence of uniform meshes: 100, 200, and 400, measure the L^1-errors, and then compute the corresponding experimental convergence rates for the density. The obtained results are presented in Table 1, where one can clearly see that the expected order of accuracy is achieved for the studied schemes. At the same time, one can see that the four low-dissipation schemes are more accurate than the HLL counterparts.

Remark 4.1. We stress that to achieve the fifth order of accuracy for the fifth-order scheme, we use smaller time steps with $\Delta t \sim (\Delta x)^{\frac{5}{3}}$ to balance the spatial and temporal errors.

4.1.2 | Example 2—Moving Contact Wave

In the second example, we consider the moving contact discontinuity problem from Reference [47] with the following initial conditions:

$$(\rho,u,p)(x,0) = \begin{cases} (1.4, 0.1, 1), & x < 0.5, \\ (1, 0.1, 1), & x > 0.5, \end{cases}$$

which is considered on the interval $[0,1]$ with the free boundary conditions at both ends.

We compute the numerical solutions until the final time $t = 0.2$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh with $\Delta x = 1/200$ and then plot them in Figure 1 together with the exact solution. As one can see, the numerical results computed by four low-dissipation schemes coincide and are better than those computed by the corresponding HLL schemes, even when they are extended to high orders.

4.1.3 | Example 3—Stationary Contact Wave, Traveling Shock and Rarefaction Wave

In the third example, we consider the strong shock interaction problem proposed in Reference [75]. The initial conditions,

$$(\rho,u,p)(x,0) = \begin{cases} (1, -19.59745, 1000) & \text{if } x < 0.8, \\ (1, -19.59745, 0.01) & \text{otherwise}, \end{cases}$$

are prescribed in the computational domain $[-5,5]$, in which free boundary conditions are implemented.

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TABLE 1 | Example 1: The L^1-errors and experimental convergence rates for the density ρ computed by the first-, second-, third-, and fifth-order schemes.

	HLL, first-order		HLL, second-order		HLL, third-order		HLL, fifth-order	
Mesh	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
100	9.91e-03	—	1.01e-03	—	2.37e-05	—	7.01e-08	—
200	4.98e-03	0.992	2.46e-04	2.04	2.96e-06	3.00	2.20e-09	5.00
400	2.50e-03	0.996	5.98e-05	2.04	3.70e-07	3.00	6.86e-11	5.00
	HLLC, first-order		**HLLC, second-order**		**HLLC, third-order**		**HLLC, fifth-order**	
Mesh	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
100	7.99e-03	—	9.40e-04	—	1.99e-05	—	5.90e-08	—
200	4.03e-03	0.990	2.24e-04	2.07	2.49e-06	3.00	1.85e-09	5.00
400	2.02e-03	0.995	5.51e-05	2.03	3.12e-07	3.00	5.78e-11	5.00
	TV, first-order		**TV, second-order**		**TV, third-order**		**TV, fifth-order**	
Mesh	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
100	7.99e-03	—	9.40e-04	—	1.99e-05	—	5.90e-08	—
200	4.03e-03	0.990	2.24e-04	2.07	2.49e-06	3.00	1.85e-09	5.00
400	2.02e-03	0.995	5.51e-05	2.03	3.12e-07	3.00	5.78e-11	5.00
	LDCU, first-order		**LDCU, second-order**		**LDCU, third-order**		**LDCU, fifth-order**	
Mesh	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
100	7.99e-03	—	9.40e-04	—	1.99e-05	—	5.90e-08	—
200	4.03e-03	0.990	2.24e-04	2.07	2.49e-06	3.00	1.85e-09	5.00
400	2.02e-03	0.995	5.51e-05	2.03	3.12e-07	3.00	5.78e-11	5.00
	LCDCU, first-order		**LCDCU, second-order**		**LCDCU, third-order**		**LCDCU, fifth-order**	
Mesh	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
100	7.99e-03	—	9.40e-04	—	1.99e-05	—	5.90e-08	—
200	4.03e-03	0.990	2.24e-04	2.07	2.49e-06	3.00	1.85e-09	5.00
400	2.02e-03	0.995	5.51e-05	2.03	3.12e-07	3.00	5.78e-11	5.00

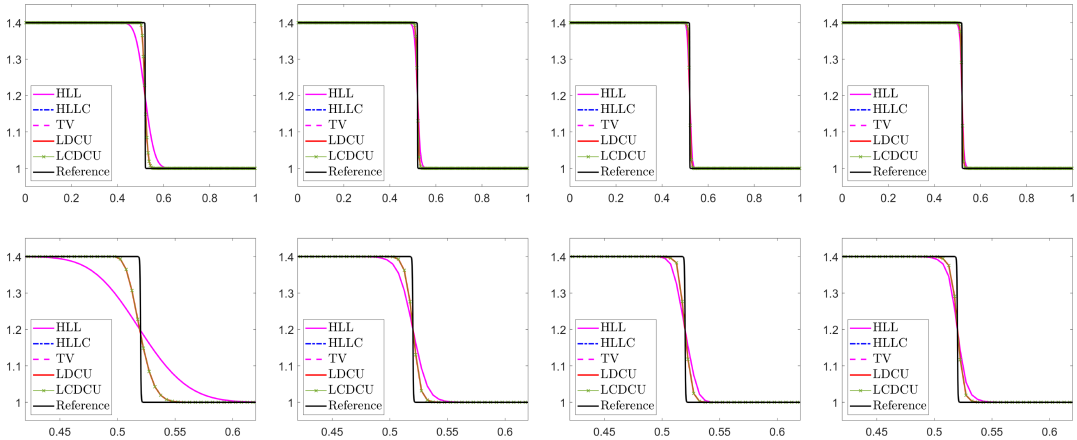


FIGURE 1 | Example 2: Density ρ computed by the first-, second-, third-, and fifth-order schemes (top row) and zoom at [0.42, 0.62] (bottom row).

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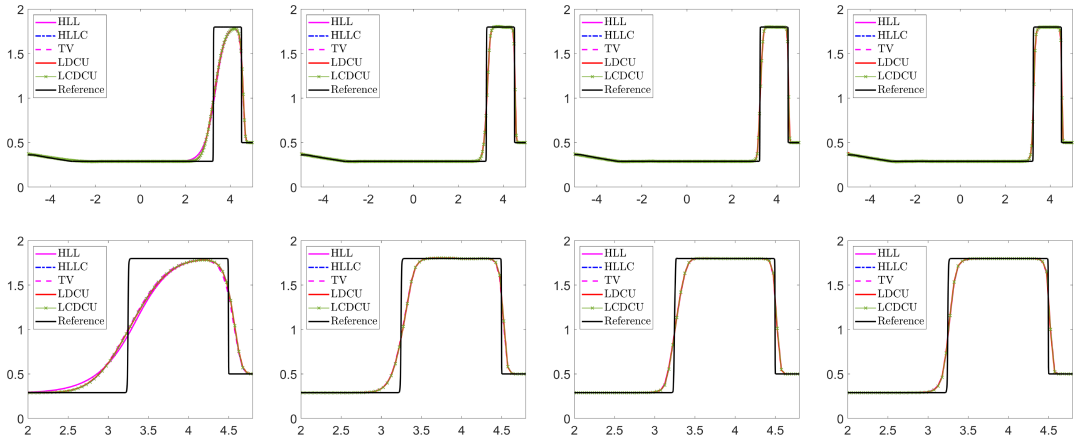


FIGURE 2 | Example 3: Density ρ computed by the first-, second-, third-, and fifth-order schemes (top row) and zoom at $[2, 4.8]$ (bottom row).

We compute the numerical solutions until the final time $t = 0.03$ using the studied first-, second-, third-, and fifth-order schemes on a uniform mesh with $\Delta x = 1/20$. The obtained numerical results are plotted in Figure 2, as well as the reference solution, which is obtained by the HLL scheme on a much finer mesh with $\Delta x = 1/400$, showing that the numerical results obtained with the four low-dissipation schemes are consistent with each other and outperform those produced by the corresponding HLL schemes, especially near the contact wave; see Figure 2 (bottom right), where we zoom at the neighborhood of the contact wave. When the studied schemes are extended to higher orders, the differences are very limited.

4.1.4 | Example 4—Shock Bubble Wave

In the fourth example, we consider the "shock-bubble" interaction problem taken from Reference [51]. The initial data, given by

$$(\rho, u, p)(x, 0) = \begin{cases} (13.1538, 0, 1) & \text{if } |x| < 0.25, \\ (1.3333, -0.3535, 1.5) & \text{if } x > 0.75, \\ (1, 0, 1) & \text{otherwise,} \end{cases}$$

correspond to a left-moving shock, initially located at $x = 0.75$, and a bubble of radius 0.25, initially located at the origin. We implement solid wall boundary conditions on the left and free boundary conditions on the right of the computational domain $[-1, 1]$.

We compute the numerical solution until the final time $t = 3$ using the studied first-, second-, third-, and fifth-order schemes on the uniform mesh with $\Delta x = 1/100$. The obtained numerical results are presented in Figure 3 along with the reference solution computed by the HLL schemes on a much finer mesh with $\Delta x = 1/2000$. One can notice that there are no clear differences in the numerical results computed by the four low-dissipation schemes, but the results computed by the low-dissipation schemes are better than the HLL scheme, especially in the first- and second-order results.

4.1.5 | Example 5—Shock-Density Wave Interaction Problem

In this example taken from [76], we consider the shock-density wave interaction problem with the following initial data,

$$(\rho, u, p)\Big|_{(x,0)} = \begin{cases} \left(\frac{27}{7}, \frac{4\sqrt{35}}{9}, \frac{31}{3}\right), & x < -4, \\ (1 + 0.2\sin(5x), 0, 1), & x > -4, \end{cases}$$

prescribed in the computational domain $[-5, 5]$ subject to the free boundary conditions.

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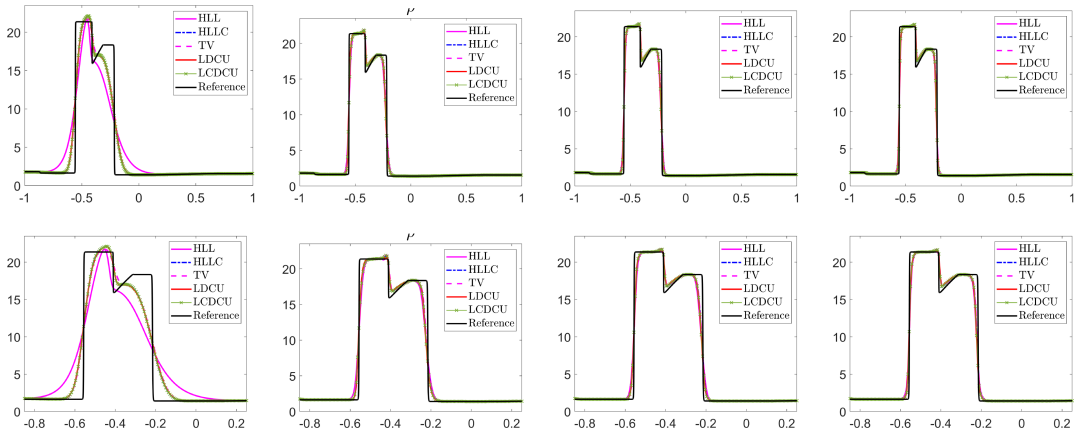


FIGURE 3 | Example 4: Density ρ computed by the first-, second-, third-, and fifth-order schemes (top row) and zoom at $[-0.85, 0.25]$ (bottom row).

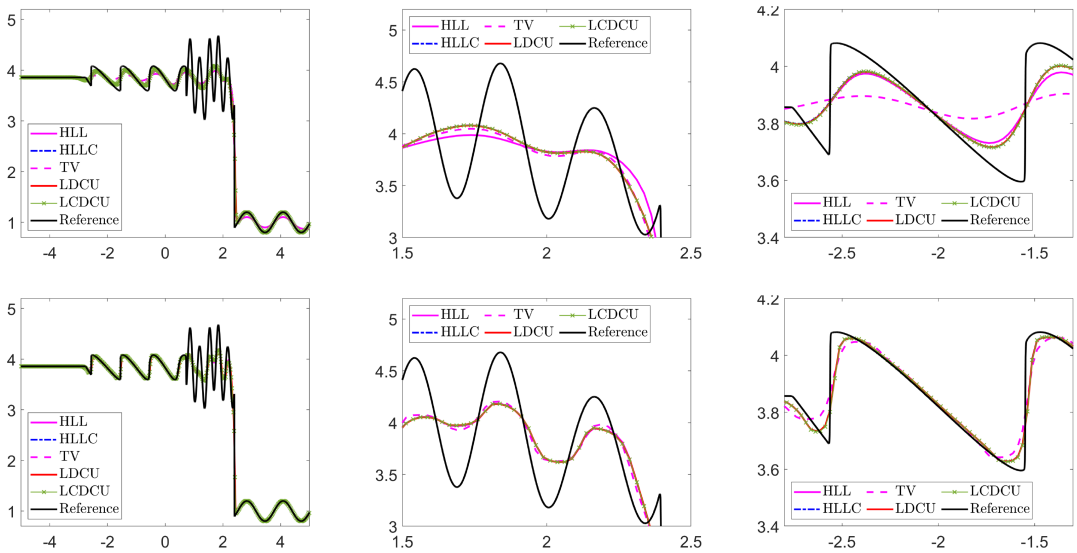


FIGURE 4 | Example 5: Density ρ computed by the first- and second-order schemes (left column) and zoom at $[1.5, 2.5]$ (middle column), and $[-2.8, -1.3]$ (right column).

We compute the numerical solutions until the final time $t = 1.8$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh of 400 cells, and present the obtained numerical results in Figures 4 and 5 together with the reference solution computed by the HLL scheme on a much finer mesh of 8000 cells. It can be seen from this example that the TV schemes exhibit higher dissipation than the other schemes, including the HLL scheme, near the shock waves even for the high-order schemes. However, the TV schemes achieve better resolution than HLL schemes in the smooth parts of the results.

4.1.6 | Example 6—Shock-Entropy Wave Interaction Problem

In this 1-D example, we consider the shock-entropy problem taken from Reference [77]. The initial conditions,

$$(\rho, u, p)(x, 0) = \begin{cases} (1.51695, 0.523346, 1.805), & x < -4.5, \\ (1 + 0.1\sin(20x), 0, 1), & x > -4.5, \end{cases}$$

correspond to a forward-facing shock wave of Mach 1.1 interacting with high-frequency density perturbations; i.e., as the shock wave moves, the perturbations spread ahead. In this example, the free boundary conditions are imposed at both ends of the computational domain $[-10, 5]$.

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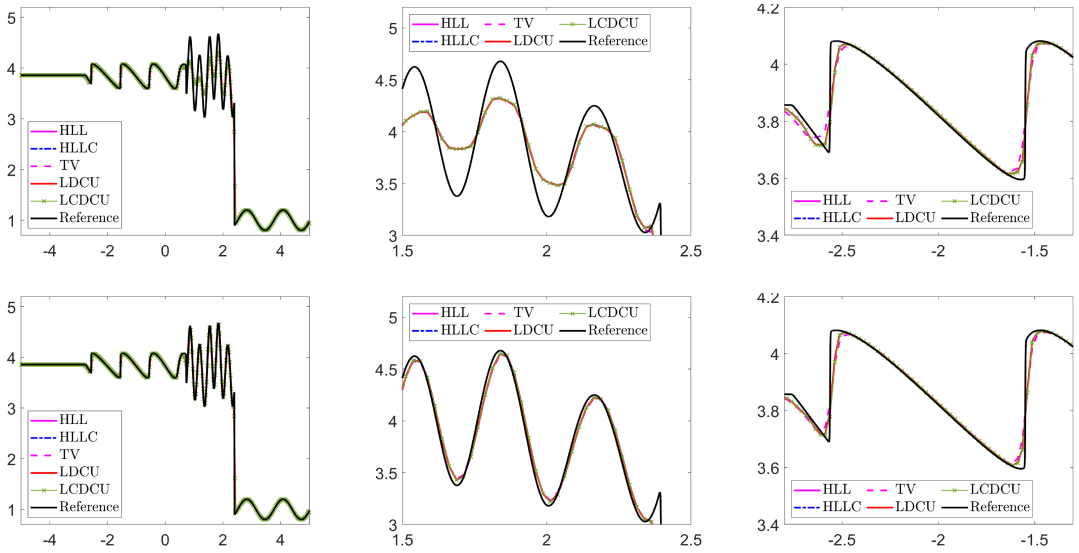


FIGURE 5 | Example 5: Density ρ computed by the third- and fifth-Order schemes (left column) and zoom at $[1.5, 2.5]$ (middle column), and $[-2.8, -1.3]$ (right column).

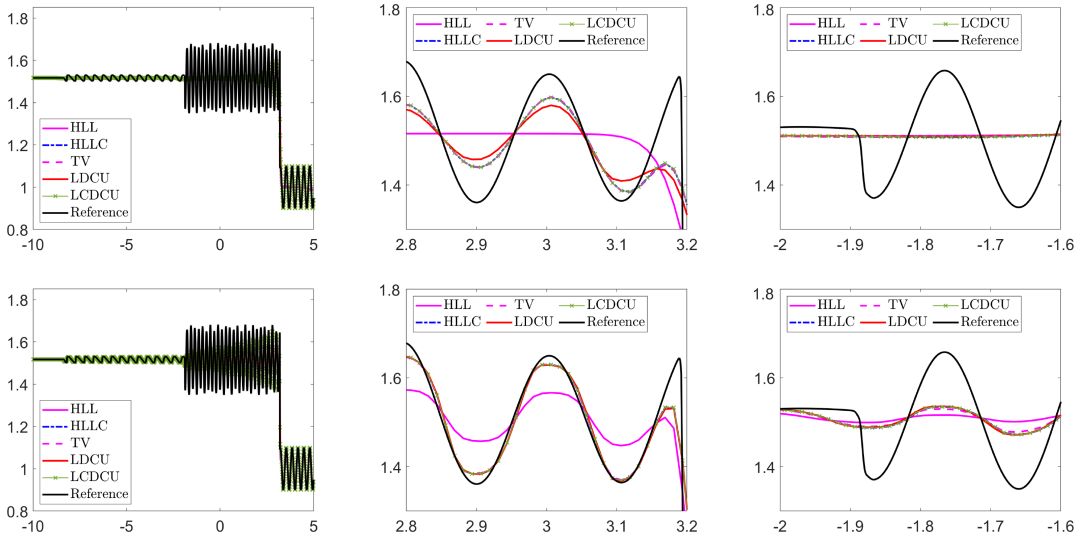


FIGURE 6 | Example 6: Density ρ computed by the first- and second-order schemes (left) and zoom at $[2.8, 3.2]$ (middle) and $[-2, -1.6]$ (right).

We compute the solutions until the final time $t = 5$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh of 1200 cells. The numerical results are shown in Figures 6–7 along with the reference solution computed by the HLL scheme on a much finer mesh of 12,000 cells. The obtained results clearly demonstrate a substantial difference in the resolution computed by HLL and the four low-dissipation schemes. One can also see that in the first-order results, the LDCU scheme has slightly more dissipation than the TV, HLLC, and LCDCU schemes, but is still significantly better than the HLL scheme. At the same time, the four low-dissipation schemes coincide when extended to higher orders; see Figures 6 (bottom) and 7.

4.1.7 | Example 7—High-Mach Moving Shock

In this example, adapted from Reference [78], we consider the one-dimensional version of a right-moving Mach-100 shock wave. The initial conditions are given by

$$(\rho, u, p)(x, 0) = \begin{cases} (5.9970015,\ 98.59147,\ 11666.5), & x < 0.2, \\ (1, 0, 1), & x > 0.2, \end{cases}$$

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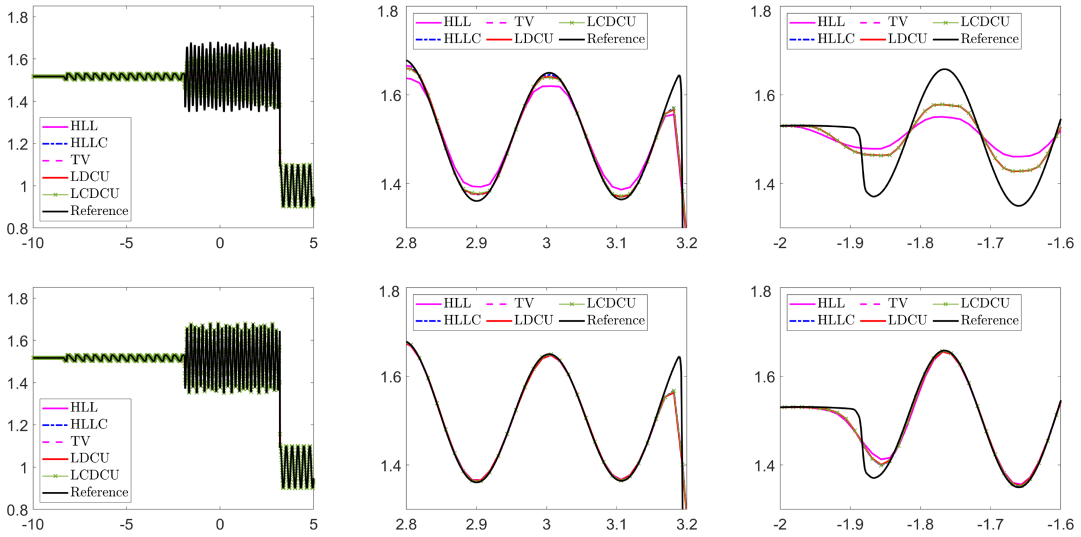


FIGURE 7 | Example 6: Density ρ computed by the third- and fifth-order schemes (left) and zoom at $[2.8, 3.2]$ (middle) and $[-2, -1.6]$ (right).

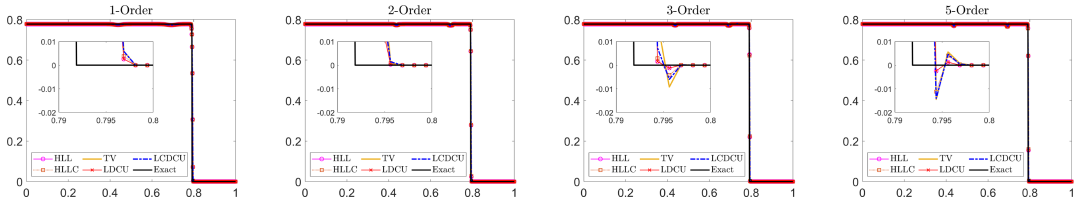


FIGURE 8 | Example 7: Decimal logarithm of the density $\log_{10}(\rho)$ computed by the HLL, HLLC, TV, LDCU, and LCDCU schemes together with the exact solution.

prescribed in the computational domain $[0, 1]$. The initial data correspond to a right-moving Mach-100 shock wave. At the left boundary, the post-shock state is imposed, while at the right boundary, the pre-shock state is imposed. More precisely, the boundary conditions are given by

$$(\rho, u, p)(0, t) = (5.9970015,\ 98.59147,\ 11666.5), \qquad (\rho, u, p)(1, t) = (1, 0, 1).$$

We compute the numerical solutions until the final time $t = 0.005$ using the studied schemes on a uniform mesh of 800 cells. The exact solution remains a single right-moving shock separating the two constant states, and its location is given by

$$x_s(0.005) \approx 0.791608.$$

The numerical results for $\log_{10}(\rho)$ are reported in Figure 8 together with the exact solution. One can see that the HLLC, TV, and LCDCU schemes exhibit spurious oscillations near the shock wave, which become more pronounced as the order increases. In contrast, the HLL and LDCU schemes do not exhibit visible numerical shock instabilities.

4.1.8 | Example 8—Colliding Strong Shocks

In the last 1-D example, motivated by the colliding-shocks problem in Reference [25], we consider the following initial conditions:

$$(\rho, u, p)(x, 0) = \begin{cases} (1, 20, 1), & x < 0, \\ (1, -20, 1), & x > 0, \end{cases}$$

which are prescribed in the computational domain $[-1, 1]$. The two initial states move toward each other and generate two strong shock waves propagating in opposite directions. At the left and right boundaries, the corresponding initial states are imposed. More precisely, the boundary conditions are given by

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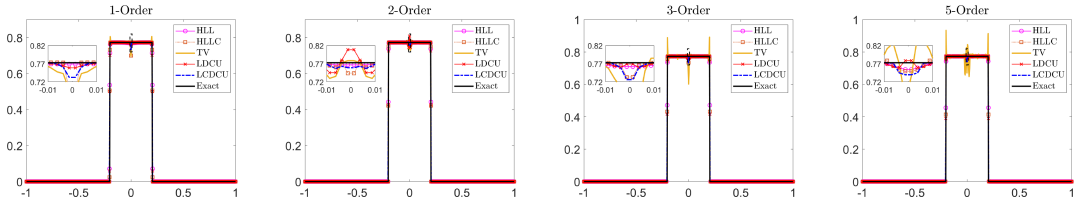


FIGURE 9 | Example 8: Decimal logarithm of the density $\log_{10}(\rho)$ computed by the HLL, HLLC, TV, LDCU, and LCDCU schemes together with the exact solution.

$$(\rho, u, p)(-1, t) = (1, 20, 1), \qquad (\rho, u, p)(1, t) = (1, -20, 1).$$

We compute the numerical solutions until the final time $t = 0.05$ using the studied schemes on a uniform mesh of 800 cells. The exact solution consists of two shocks separating the left and right states from a constant intermediate state:

$$(\rho, u, p)(x, t) = \begin{cases} (1, 20, 1), & x < S_L t, \\ (\rho_*, 0, p_*), & S_L t < x < S_R t, \\ (1, -20, 1), & x > S_R t, \end{cases}$$

where

$$\rho_* \approx 5.9283027607, \qquad p_* \approx 482.1638447197,$$

and the speeds of the left- and right-moving shocks are

$$S_L \approx -4.0581922360, \qquad S_R \approx 4.0581922360.$$

Therefore, at the final time $t = 0.05$, the two shocks are located at

$$x_L = S_L t \approx -0.202909612, \qquad x_R = S_R t \approx 0.202909612.$$

The numerical results for $\log_{10}(\rho)$ are reported in Figure 9 together with the exact solution. The four low-dissipation schemes exhibit more pronounced spurious oscillations than the corresponding HLL schemes. Among them, the TV schemes produce the strongest oscillations, especially when extended to the third and fifth orders.

4.2 | Two-Dimensional Examples

In this section, we consider the 2-D Euler equations of gas dynamics (1.2), (3.1) and (3.2). In Examples 9–18, we take the specific heat ratio $\gamma = 1.4$, while in Example 19, we take $\gamma = 5/3$.

4.2.1 | Example 9—2-D Accuracy Test

In the first 2-D example taken from References [55, 79], we consider the 2-D Euler equations of gas dynamics subject to the periodic initial conditions,

$$\rho(x, y, 0) = 1 + \frac{1}{5}\sin(\pi(x + y)), \quad u(x, y, 0) \equiv 1, \quad v(x, y, 0) \equiv -0.7, \quad p(x, y, 0) \equiv 1,$$

prescribed on $[-1, 1] \times [-1, 1]$. The exact solution of this initial value problem can be easily obtained and is given by

$$\rho(x, y, t) = 1 + \frac{1}{5}\sin[\pi(x + y - 0.3t)], \quad u(x, y, t) \equiv 1, \quad v(x, y, t) \equiv -0.7, \quad p(x, y, t) \equiv 1.$$

We first compute the numerical solution until the final time $t = 0.1$ using the first-, second-, third-, and fifth-order schemes on a sequence of uniform meshes: 100×100, 200×200, and 400×400, and then measure the L^1-errors and the corresponding experimental convergence rates for the density. The obtained results are presented in Table 2, where one can see that the studied first-, second-, third-, and fifth-order schemes achieve the expected order of accuracy.

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TABLE 2 | Example 9: The L^1-errors and experimental convergence rates for the density ρ computed by the first-, second-, third-, and fifth-order schemes.

		First-order		Second-order		Third-order		Fifth-order	
Method	**Mesh**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
HLL	100×100	1.18e-02	—	3.59e-04	—	6.02e-06	—	4.36e-09	—
	200×200	5.91e-03	0.994	8.38e-05	2.10	7.37e-07	3.03	1.36e-10	5.00
	400×400	2.96e-03	0.997	1.91e-05	2.13	9.22e-08	3.00	4.42e-12	4.95
HLLC	100×100	8.39e-03	—	2.64e-04	—	4.32e-06	—	3.12e-09	—
	200×200	4.21e-03	0.994	6.16e-05	2.10	5.27e-07	3.03	9.76e-11	5.00
	400×400	2.11e-03	0.997	1.47e-05	2.06	6.59e-08	3.00	3.19e-12	4.93
TV	100×100	8.39e-03	—	2.64e-04	—	4.32e-06	—	3.12e-09	—
	200×200	4.21e-03	0.994	6.16e-05	2.10	5.27e-07	3.03	9.76e-11	5.00
	400×400	2.11e-03	0.997	1.47e-05	2.06	6.59e-08	3.00	3.19e-12	4.93
LDCU	100×100	8.39e-03	—	2.64e-04	—	4.32e-06	—	3.12e-09	—
	200×200	4.21e-03	0.994	6.16e-05	2.10	5.27e-07	3.03	9.76e-11	5.00
	400×400	2.11e-03	0.997	1.47e-05	2.06	6.59e-08	3.00	3.19e-12	4.93
LCDCU	100×100	8.39e-03	—	2.64e-04	—	4.32e-06	—	3.12e-09	—
	200×200	4.21e-03	0.994	6.16e-05	2.10	5.27e-07	3.03	9.76e-11	5.00
	400×400	2.11e-03	0.997	1.47e-05	2.06	6.59e-08	3.00	3.19e-12	4.93

At the same time, the four low-dissipation schemes are slightly more accurate than the HLL schemes, while the four low-dissipation schemes achieve exactly the same numerical results. As in Example 1, we had to use smaller time steps with $\Delta t \sim \min\left\{(\Delta x)^{\frac{5}{3}}, (\Delta y)^{\frac{5}{3}}\right\}$ to achieve the fifth order of accuracy.

4.2.2 | Example 10—2-D Vortex Evolution Problem

In this example, taken from Reference [80]; see also [33, 81], we consider the 2-D vortex evolution problem with the following initial conditions

$$(\rho(x,y,0), u(x,y,0), v(x,y,0), p(x,y,0)) = \left(T^{\frac{1}{\gamma-1}}, 1-\frac{\varepsilon}{2\pi}e^{\frac{1}{2}(1-r^2)}y, 1+\frac{\varepsilon}{2\pi}e^{\frac{1}{2}(1-r^2)}x, \rho^{\gamma}\right),$$

where $T = 1-\frac{(\gamma-1)\varepsilon^2}{8\gamma\pi^2}e^{(1-r^2)}$, $r^2 = x^2+y^2$, and $\varepsilon = 5$ is the vortex strength. The initial data, prescribed in the computational domain $[-5,5]\times[-5,5]$ subject to the periodic boundary conditions, corresponds to a smooth vortex placed at the origin and is defined as the isentropic perturbation to the uniform flow of unit values of primitive variables and the exact solution is a vortex moving with a constant velocity at 45° to the Cartesian mesh lines.

We compute the numerical solution until the final time $t = 10$ using the first-, second-, third-, and fifth-Order schemes on a sequence of uniform meshes: 100×100, 200×200, and 400×400, and then measure the L^1-errors between the computed solutions and the exact solutions and the corresponding experimental convergence rates for the density. The obtained results are presented in Table 3, where one can see that both first-order and second-order convergence rates are observed only after significant mesh refinement, while the third- and fifth-order schemes achieve the expected order of accuracy. To have a better view, we also show the L^1-errors in Figure 10, where we also show the results computed by the first- and second-order schemes on finer meshes 800×800 and 1600×1600 to show that the first- and second-order schemes achieve expected convergence rates after mesh refinement. It is noticed that, in this example, the TV schemes are slightly more dissipative than the HLL schemes, but HLLC, LDCU, and LCDCU schemes are slightly more accurate than the HLL schemes.

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TABLE 3 | Example 10: L^1-errors and experimental convergence rates for the density ρ computed by the first-, second-, third-, and fifth-order HLL, HLLC, TV, LDCU, and LCDCU schemes.

		First-order		Second-order		Third-order		Fifth-order	
Method	**Mesh**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
HLL	100×100	1.86	—	1.48e-01	—	2.45e-02	—	7.88e-04	—
	200×200	1.33	4.84e-01	4.54e-02	1.71	2.94e-03	3.06	2.42e-05	5.02
	400×400	0.84	6.71e-01	1.00e-02	2.18	3.08e-04	3.25	6.16e-07	5.30
HLLC	100×100	1.74	—	1.44e-01	—	2.26e-02	—	7.03e-04	—
	200×200	1.21	5.26e-01	4.67e-02	1.62	2.68e-03	3.08	2.27e-05	4.95
	400×400	0.74	7.04e-01	9.88e-03	2.24	2.70e-04	3.31	5.60e-07	5.34
LDCU	100×100	1.76	—	1.54e-01	—	2.23e-02	—	7.07e-04	—
	200×200	1.23	5.21e-01	5.07e-02	1.60	2.71e-03	3.04	2.32e-05	4.93
	400×400	0.76	6.97e-01	1.05e-02	2.28	2.85e-04	3.25	5.94e-07	5.29
LCDCU	100×100	1.74	—	1.44e-01	—	2.28e-02	—	7.03e-04	—
	200×200	1.21	5.26e-01	4.69e-02	1.62	2.70e-03	3.07	2.27e-05	4.95
	400×400	0.74	7.04e-01	9.88e-03	2.25	2.70e-04	3.32	5.60e-07	5.34
TV	100×100	2.49	—	1.24e-01	—	2.98e-02	—	9.23e-04	—
	200×200	1.91	3.84e-01	3.00e-02	2.05	3.58e-03	3.06	2.49e-05	5.21
	400×400	1.26	5.97e-01	6.75e-03	2.15	3.99e-04	3.16	6.62e-07	5.23

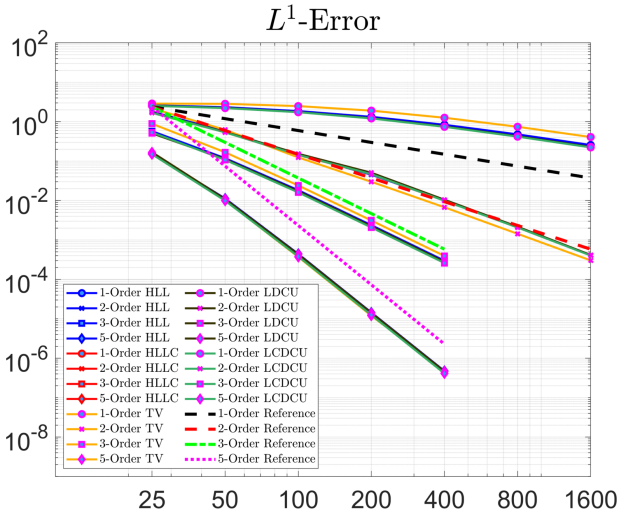


FIGURE 10 | Example 10: L^1-errors for the density ρ computed by the first-, second-, third-, and fifth-order HLL, HLLC, TV, LDCU, and LCDCU schemes.

4.2.3 | Example 11—2-D Moving Contact Waves

In this example taken from Reference [48] (see also [51, 52]), we consider an isolated moving contact wave with the following initial data:

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (1.4, 0, 0.2, 1), & (x, y) \in D \\ (1, 0, 0.2, 1), & \text{otherwise,} \end{cases}$$

where the domain D consists of the points (x, y) satisfying the following conditions:

$$\{-0.1 < x < 0.1,\ 0 < y < 0.02\} \cup \{-0.02 < x < 0.02,\ 0.02 < y < 0.1\} \cup$$
$$\{(x + 0.02)^2 + (y - 0.02)^2 < 0.08^2\} \cup \{(x - 0.02)^2 + (y - 0.02)^2 < 0.08^2\}.$$

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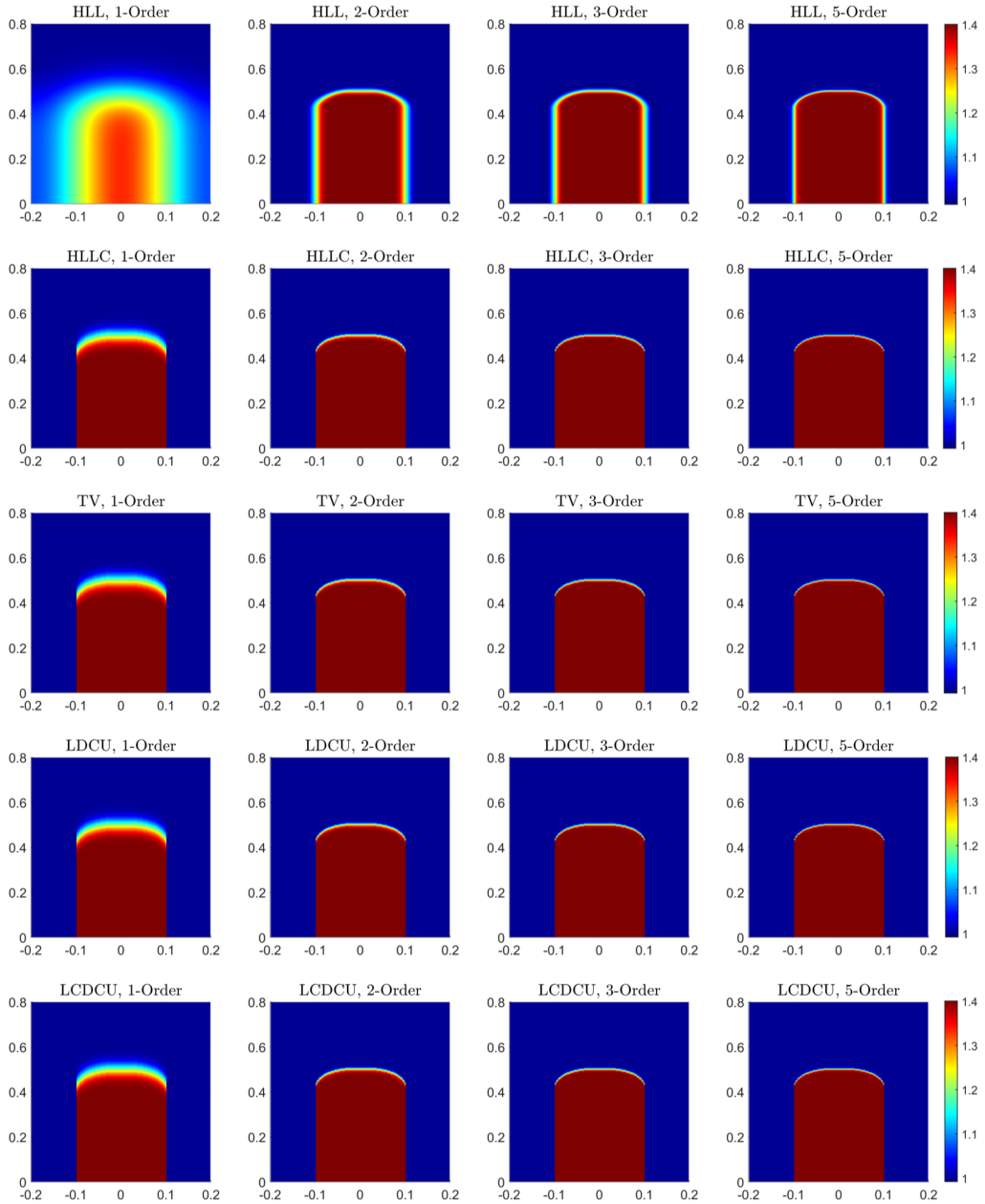


FIGURE 11 | Example 11: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

The initial data are prescribed in the computational domain $[-0.2, 0.2] \times [0, 0.8]$ subject to the free boundary conditions.

We compute the numerical solutions until the final time $t = 2$ using the studied first-, second-, third-, and fifth-Order schemes on a uniform mesh with $\Delta x = \Delta y = 1/400$ and plot the obtained results in Figure 11. As one can see, the four low-dissipation schemes produce nearly identical numerical results, which are significantly sharper than those computed by the HLL schemes, especially for the non-moving jumps across the lines $x = \pm 0.1$.

4.2.4 | Example 12—Explosion Problem

In this example, we consider the explosion problem studied in References [9, 82] (see also [51, 70]). We take the following initial conditions:

$$(\rho(x, y, 0), u(x, y, 0), v(x, y, 0), p(x, y, 0)) = \begin{cases} (1, 0, 0, 1), & x^2 + y^2 < 0.16, \\ (0.125, 0, 0, 0.1), & \text{otherwise}, \end{cases}$$

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which are prescribed in the computational domain $[-1, 1] \times [-1, 1]$, subject to free boundary conditions at all four sides. The solution of this initial-boundary value problem develops circular shock and contact waves. While the shock wave is stable and requires sufficient dissipation for stable capturing, the contact wave is unstable and can only be stabilized by numerical diffusion. This makes the problem a useful benchmark for evaluating the dissipation of numerical schemes, where the goal is to minimize dissipation while retaining shock stability.

We apply the studied first-, second-, third-, and fifth-order schemes and compute the numerical solutions until the final time $t = 3.2$ on a uniform mesh with $\Delta x = \Delta y = 3/800$. The obtained results are presented in Figure 12, where one can see that the differences between the numerical results computed by the four low-dissipation schemes are limited, but much more substantially "curlier" and the mixing layer is slightly wider (indicating a more severe instability) than those computed by the HLL schemes, while the shock is still stable.

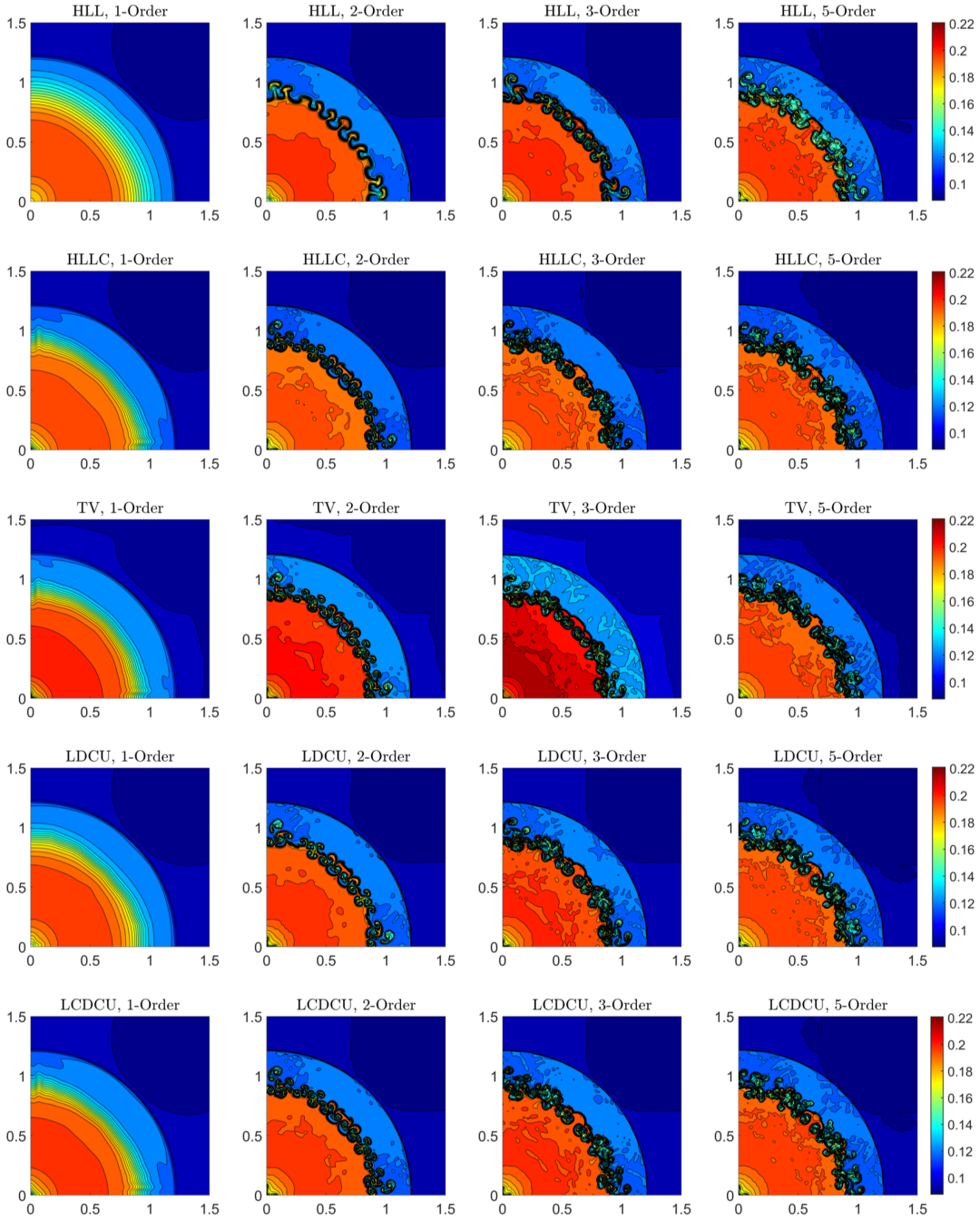


FIGURE 12 | Example 12: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

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4.2.5 | Example 13—Implosion Problem

In this example, we consider the implosion problem taken from Reference [82] (see also [47, 55]). The initial conditions

$$(\rho(x,y,0),u(x,y,0),v(x,y,0),p(x,y,0))=\begin{cases}(0.125,0,0,0.14), & |x|+|y|<0.15,\\ (1,0,0,1), & \text{otherwise},\end{cases}$$

are prescribed in the computational domain $[0,0.3]\times[0,0.3]$ with solid boundary conditions imposed at all four sides. This example is designed to assess the numerical diffusion of different schemes: A jet forms near the origin and propagates along the diagonal $y=x$, and excessive diffusion may either smear the jet entirely or alter its propagation velocity.

We compute the numerical solutions until the final time $t=2.5$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh with $\Delta x=\Delta y=3/4000$ and plot the obtained results in Figure 13, where one can clearly observe that

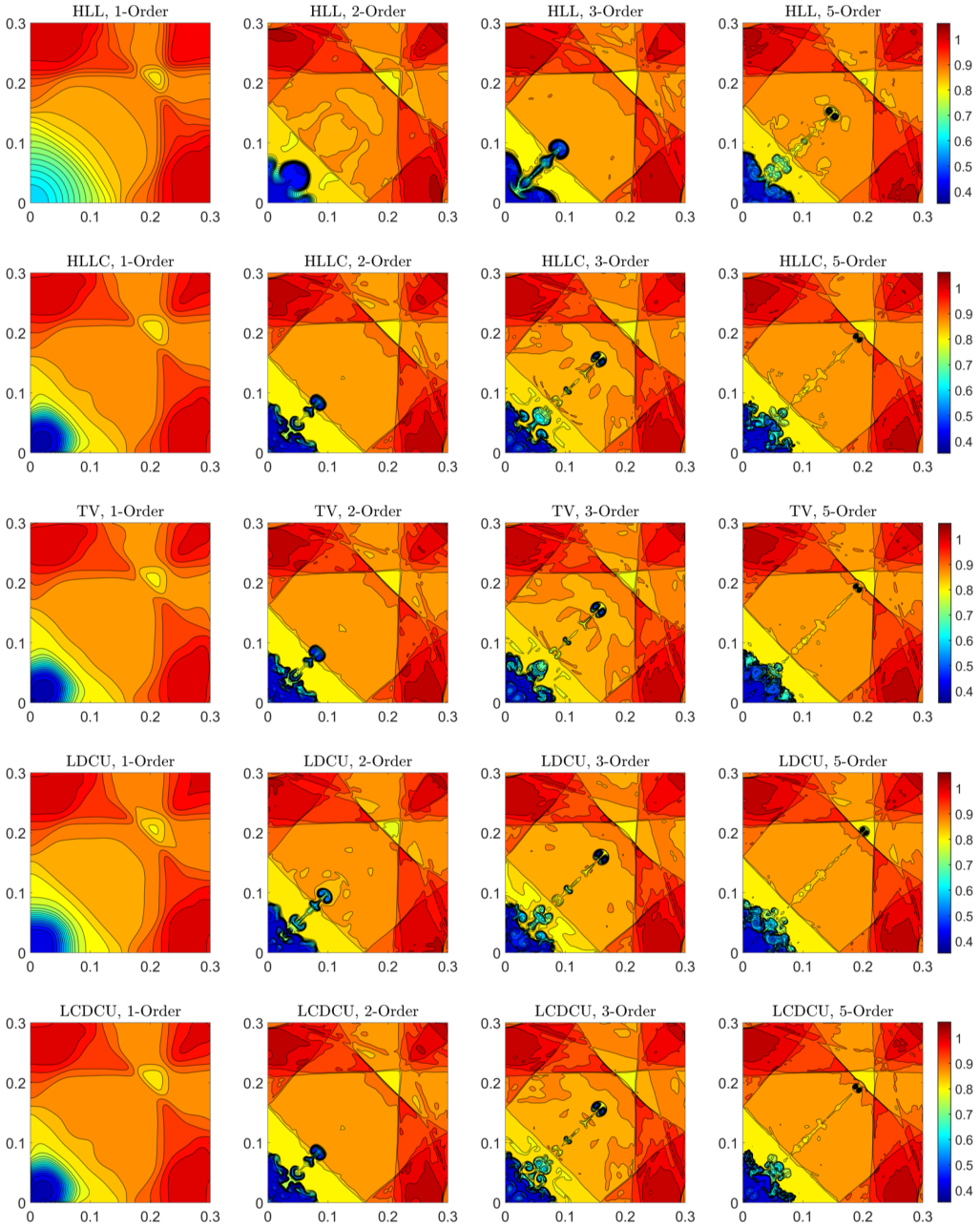


FIGURE 13 | Example 13: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

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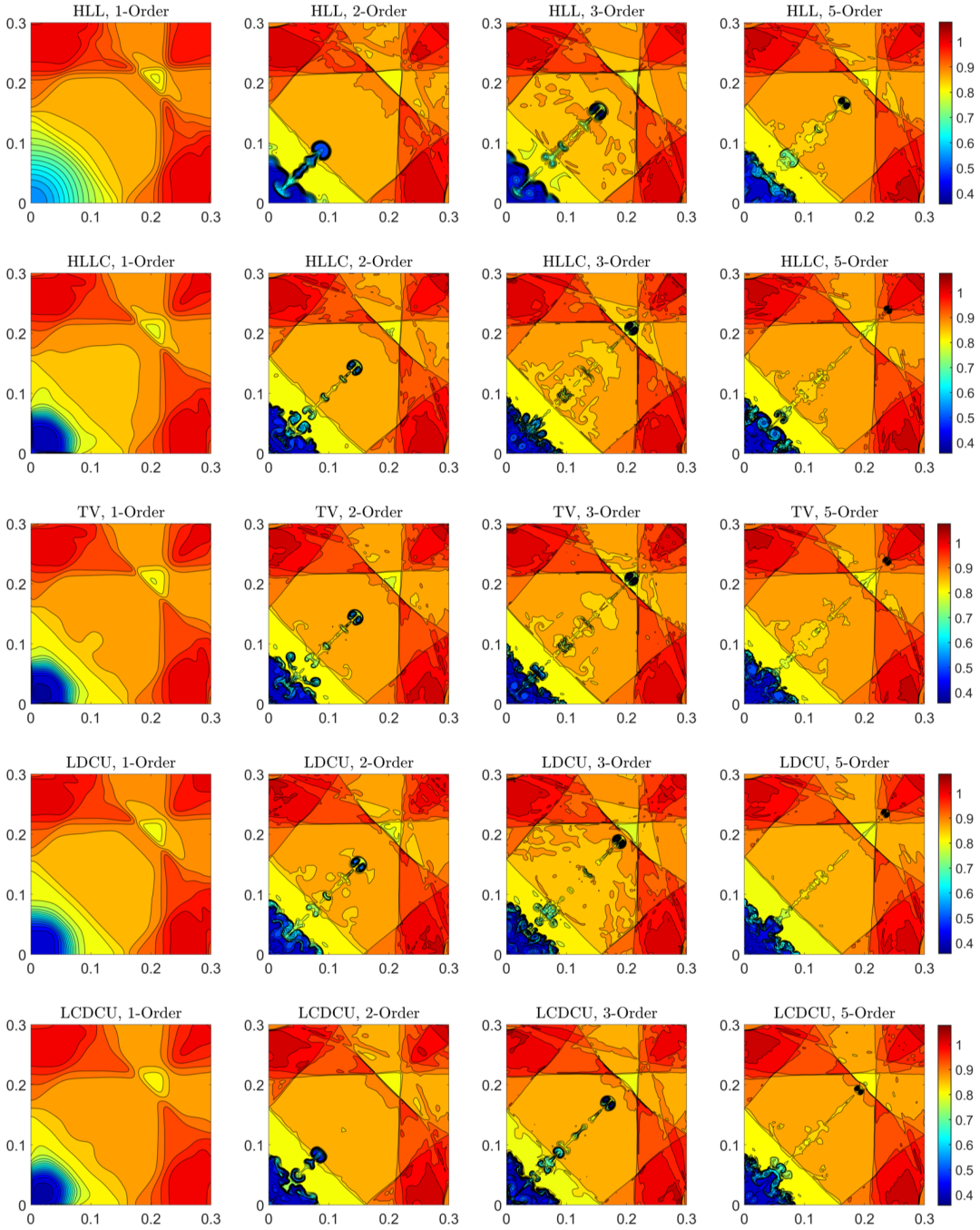


FIGURE 14 | Example 13: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

the jet propagates much further in the diagonal direction when using the four low-dissipation schemes. At the same time, the LDCU scheme is slightly less dissipative than the other three low-dissipation counterparts.

It is instructive to check the efficiency of the studied schemes. To this end, we measure the CPU time consumed by the first-, second-, third-, and fifth-order LCDCU schemes and refine the mesh used by the other four schemes to the level that exactly the same CPU time is consumed to compute all four numerical solutions. The corresponding meshes are 400×400 for the first-, second-, third-, and fifth-order LCDCU schemes, 570×570, 568×568, 482×482, and 462×462 for the HLL schemes, 568×568, 565×565, 479×479, and 458×458 for the HLLC schemes, 591×591, 590×590, 487×487, and 466×466 for the TV schemes, and 565×565, 563×563, 475×475, and 455×455 for the LDCU schemes. The obtained numerical results, presented in Figure 14, indicate that the four low-dissipation schemes are more efficient than the HLL counterparts. At the same time, it is noticed that the positions of the jets produced by the HLLC and TV

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schemes are consistent and are slightly further than those produced by the LDCU schemes. The LCDCU schemes are more computationally expensive than the other three low-dissipation schemes, and the reason is that the LCDCU fluxes defined in Subsection 3.5 are more complicated than the others.

4.2.6 | Example 14—2-D Riemann Problem (Configuration 3)

In this example, we consider Configuration 3 of the 2-D Riemann problems from Reference [83] (see also [84–86]) with the following initial conditions:

$$(\rho,u,v,p)(x,y,0)=\begin{cases}(1.5,0,0,1.5), & x>1,\ y>1,\\ (0.5323,1.206,0,0.3), & x<1,\ y>1,\\ (0.138,1.206,1.206,0.029), & x<1,\ y<1,\\ (0.5323,0,1.206,0.3), & x>1,\ y<1.\end{cases}$$

which are prescribed in the computational domain $[0,1.2]\times[0,1.2]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t=1$ by the first-, second-, third-, and fifth-order schemes on a uniform mesh with $\Delta x=\Delta y=6/5000$ and plot the obtained results in Figure 15, where one can see that the HLLC, LDCU, and LCDCU schemes resolve more small-scale structures than the HLL scheme in capturing a sideband instability of the jet in the zones of strong along-jet velocity shear and the instability along the jet's neck. However, one can see that the TV schemes can capture more details of the instability along the jet's neck, but there are some oscillations in the numerical results. At the same time, the third- and fifth-order TV schemes fail in this simulation because numerical oscillations lead to negative pressure.

4.2.7 | Example 15—2-D Riemann Problem (Configuration 6)

In this example, we consider Configuration 6 of the 2-D Riemann problems from Reference [83] (see also [84–86]) with the following initial conditions:

$$(\rho,u,v,p)\Big|_{(x,y,0)}=\begin{cases}(1,0.75,-0.5,1), & x>0.5,\ y>0.5,\\ (2,0.75,0.5,1), & x<0.5,\ y>0.5,\\ (1,-0.75,0.5,1), & x<0.5,\ y<0.5,\\ (3,-0.75,-0.5,1), & x>0.5,\ y<0.5,\end{cases}$$

which are prescribed in the computational domain $[0,1]\times[0,1]$ subject to the free boundary conditions.

We compute the numerical solution until the final time $t=1$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh with $\Delta x=\Delta y=1/600$ and plot the obtained results in Figure 16, where one can see that the four low-dissipation schemes produce more intricate vortex structures than the HLL schemes, consistent with their lower numerical dissipation.

4.2.8 | Example 16—Kelvin-Helmholtz (KH) Instability

In this example, we study the KH instability taken from References [87, 88] (see also [55, 89]). We take the following initial data:

$$(\rho(x,y,0),u(x,y,0))=\begin{cases}(1,-0.5+0.5e^{(y+0.25)/L}), & y<-0.25,\\ (2,0.5-0.5e^{(-y-0.25)/L}), & -0.25<y<0,\\ (2,0.5-0.5e^{(y-0.25)/L}), & 0<y<0.25,\\ (1,-0.5+0.5e^{(0.25-y)/L}), & y>0.25,\end{cases}$$

$$v(x,y,0)=0.01\sin(4\pi x),\qquad p(x,y,0)\equiv 1.5,$$

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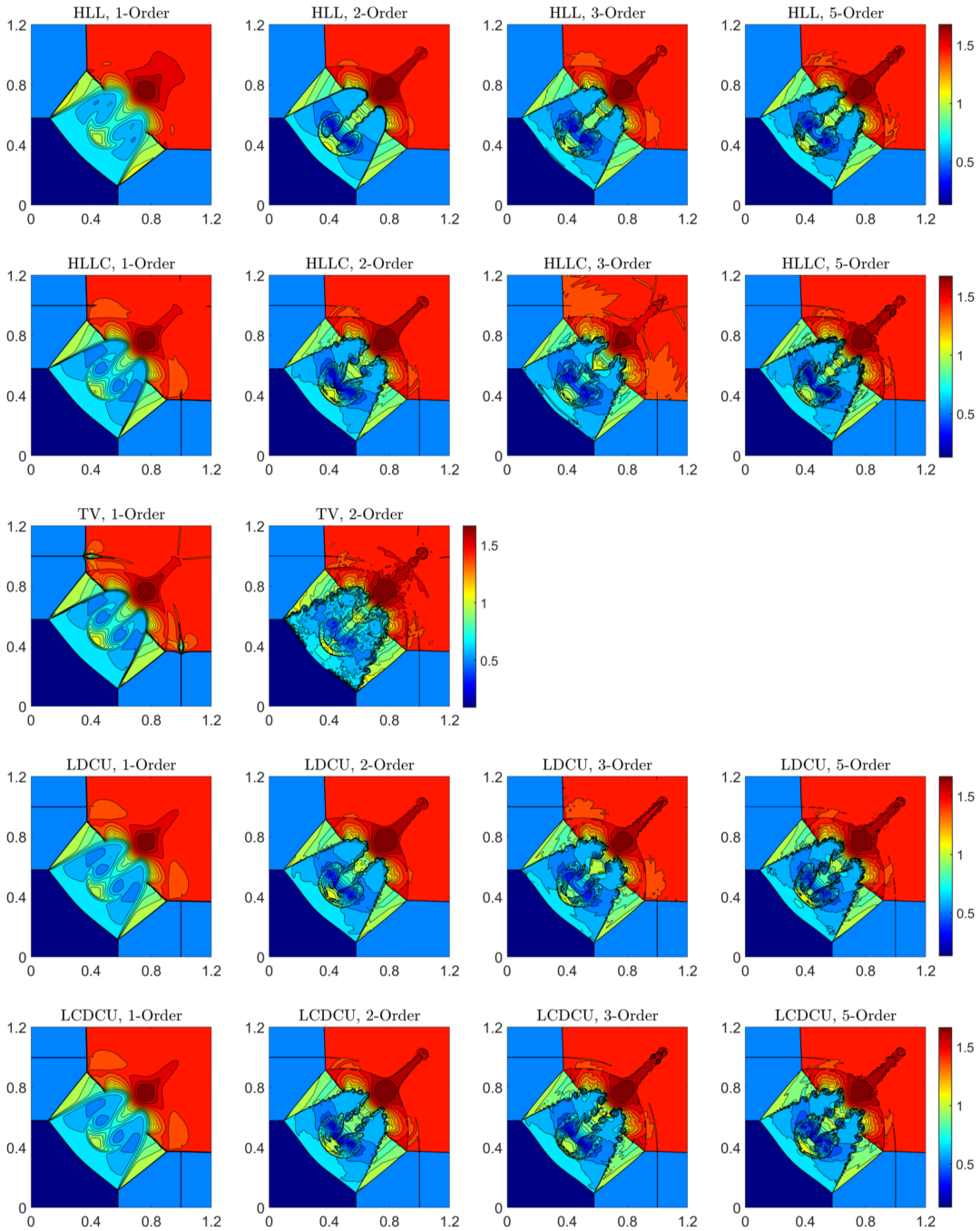


FIGURE 15 | Example 14: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row; the third- and fifth-order results are omitted because of loss of positivity), LDCU (fourth row), and LCDCU (bottom row) schemes.

where L is a smoothing parameter (we take $L = 0.00625$), which corresponds to a thin shear interface with a perturbed vertical velocity field v in the conducted simulations. The periodic boundary conditions are imposed on all four sides of the computational domain $[-0.5, 0.5] \times [-0.5, 0.5]$.

We compute the numerical solutions until the final time $t = 4$ using the first-, second-, third-, and fifth-order schemes on a uniform mesh of 1024×1024 cells, and plot the numerical results at times $t = 1$, 2.5, and 4 in Figures 17–19. One can observe that at the early time $t = 1$, the vortex streets generated by the high-order schemes are more pronounced; see Figure 17. These structures develop over time, leading to increasingly complex vortical patterns, particularly at the later times $t = 2.5$ and 4; see Figures 18 and 19. At the same time, it is clear that the four low-dissipation schemes produce more small-scale vortical structures than the HLL schemes.

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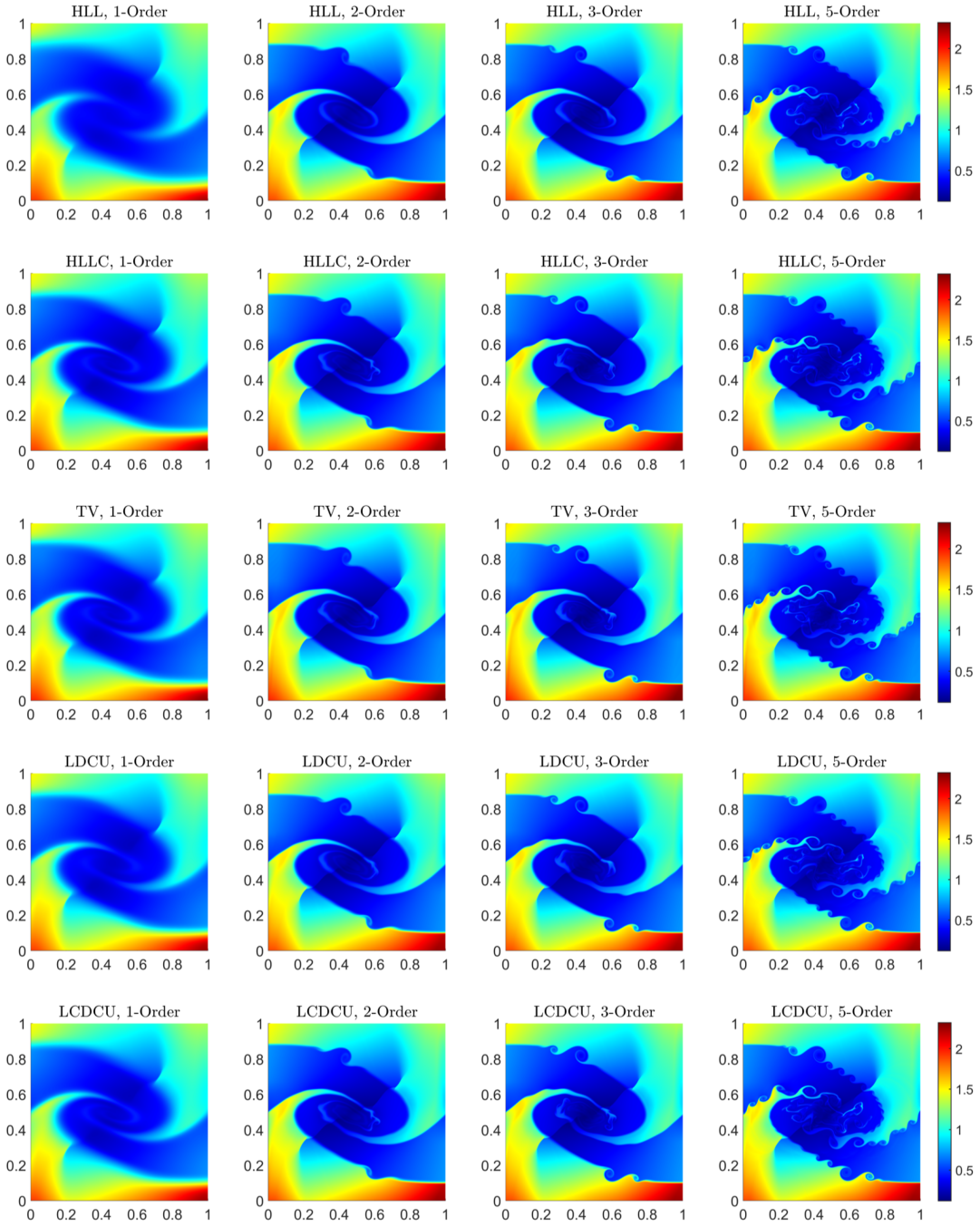


FIGURE 16 | Example 15: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

4.2.9 | Example 17—Quirk's Odd–Even Decoupling Problem

In this example, we consider a variant of Quirk's odd–even decoupling problem [90], which is designed to examine the stability of numerical schemes in the presence of a strong grid-aligned shock. Multidimensional shock instabilities, including odd–even decoupling and the carbuncle phenomenon, have also been analyzed in connection with the dissipative terms contained in numerical mass fluxes; see, e.g., [58, 59]. A right-moving Mach-6 shock is initially located near $x = 0.5$. The initial conditions are given by

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (7.376,\ 4.861,\ 0,\ 41.833), & x < x_s(y), \\ (1.4,\ 0,\ 0,\ 1), & x > x_s(y), \end{cases}$$

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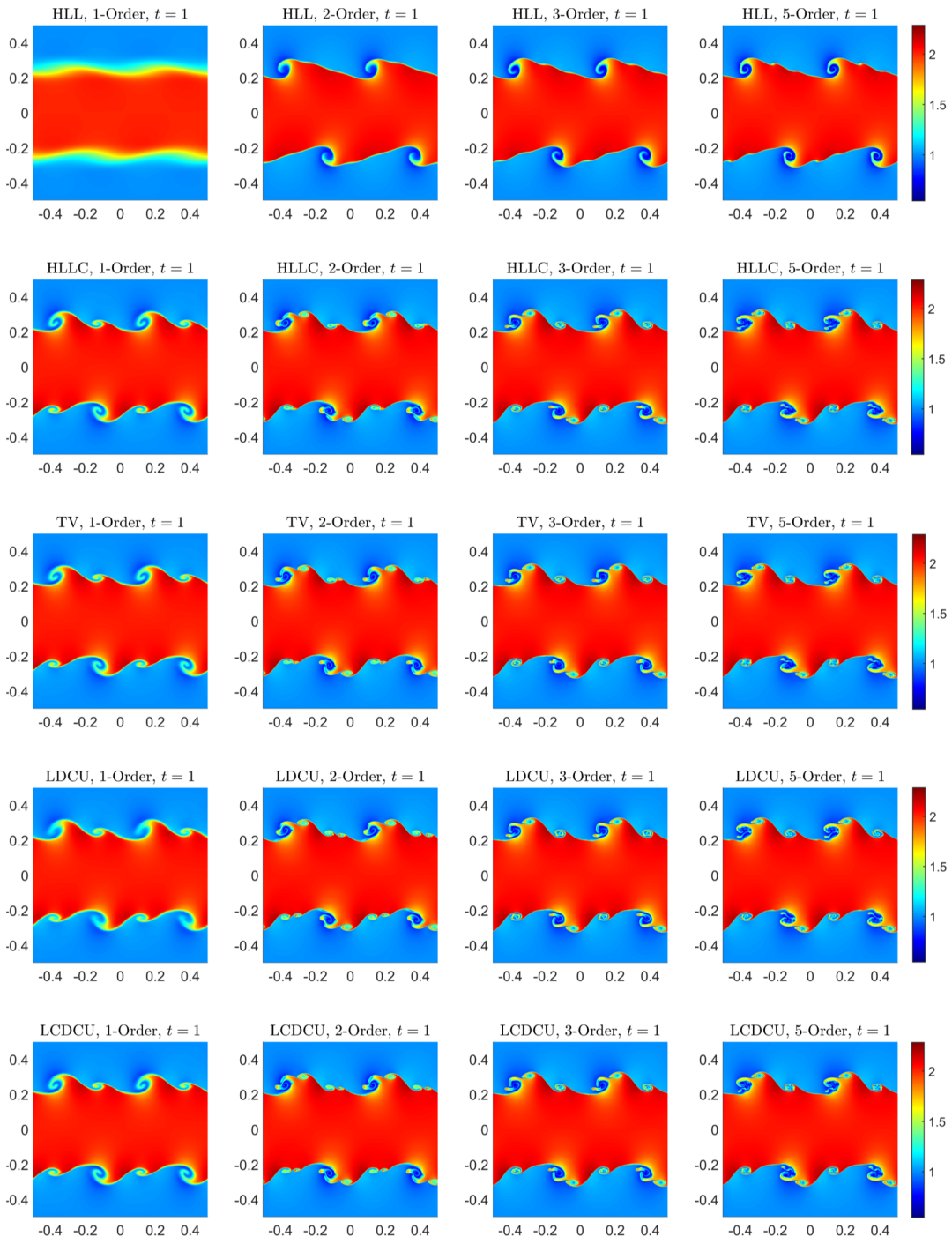


FIGURE 17 | Example 16: Time snapshots of density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes at $t = 1$.

where the left and right states are the post-shock and pre-shock states, respectively.

To trigger the odd–even decoupling instability, we introduce a small row-dependent perturbation to the initial shock location. More precisely, for

$$y_k = \left(k - \frac{1}{2}\right)\Delta y, \qquad \kappa_k = \min\{k, N_y + 1 - k\},$$

the shock location is prescribed as

$$x_s(y_k) = x_0 + (-1)^{\kappa_k}\varepsilon, \qquad x_0 = 0.5 + \frac{\Delta x}{2}, \qquad \varepsilon = 10^{-2}\Delta x.$$

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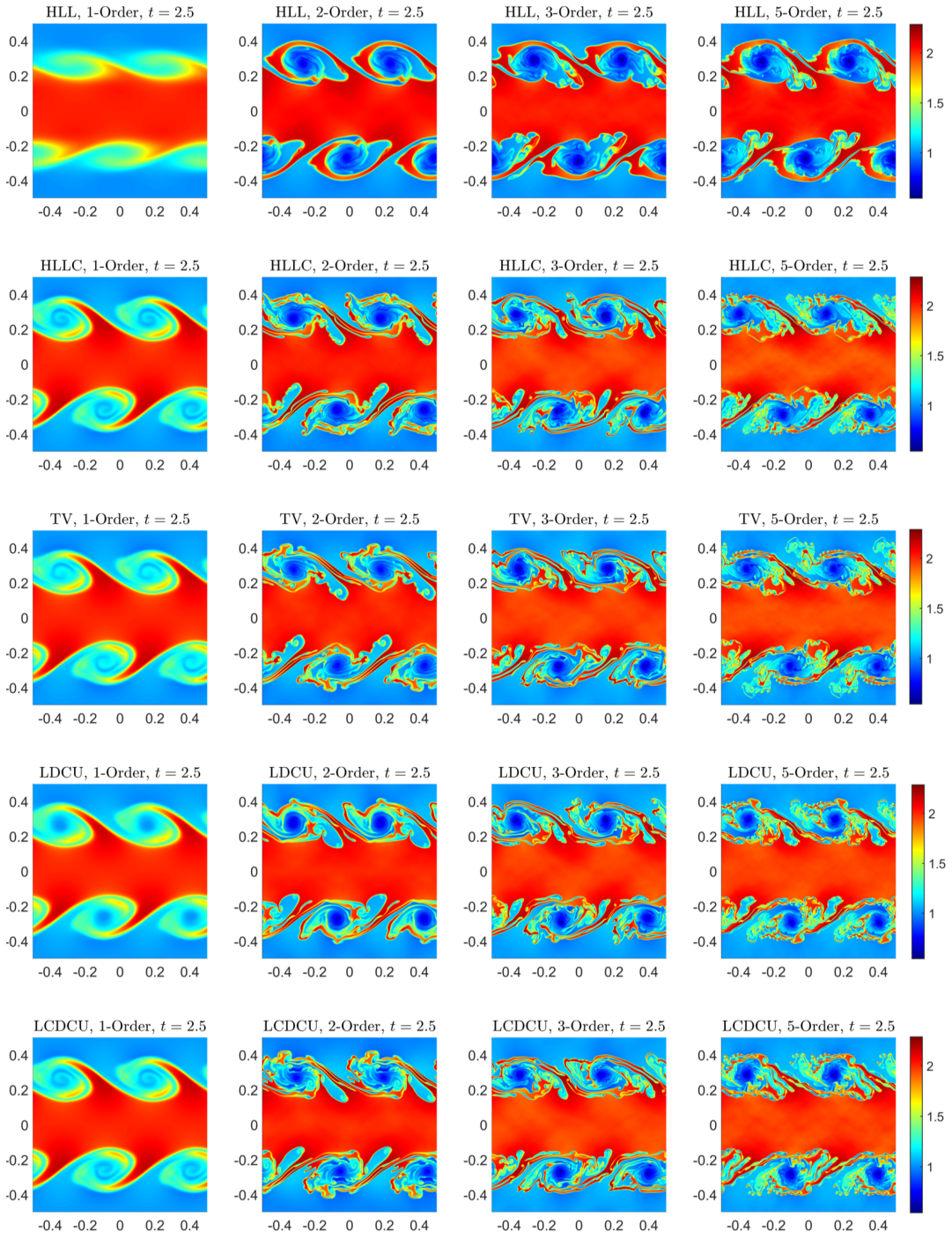


FIGURE 18 | Example 16: Time snapshots of density $[\rho]$ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes at $[t = 2.5]$.

This choice makes the perturbation symmetric with respect to the channel centerline $y = 0.5$. The initial data are prescribed in the computational domain $[0, 8] \times [0, 1]$. At the left boundary, the post-shock state is imposed, while a free boundary condition is implemented at the right boundary. Solid-wall boundary conditions are imposed at both the bottom and top boundaries.

We compute the numerical solutions until the final time $t = 1$ using the studied schemes on a uniform mesh with $\Delta x = \Delta y = 1/40$ (with the CFL number 0.3 and 0.1 for the third- and fifth-order schemes, respectively, to reduce the risk of loss of positivity). The unperturbed shock propagates at the speed $s = 6$ and is therefore located approximately at

$$x_s(1) = x_0 + 6 \approx 6.5125$$

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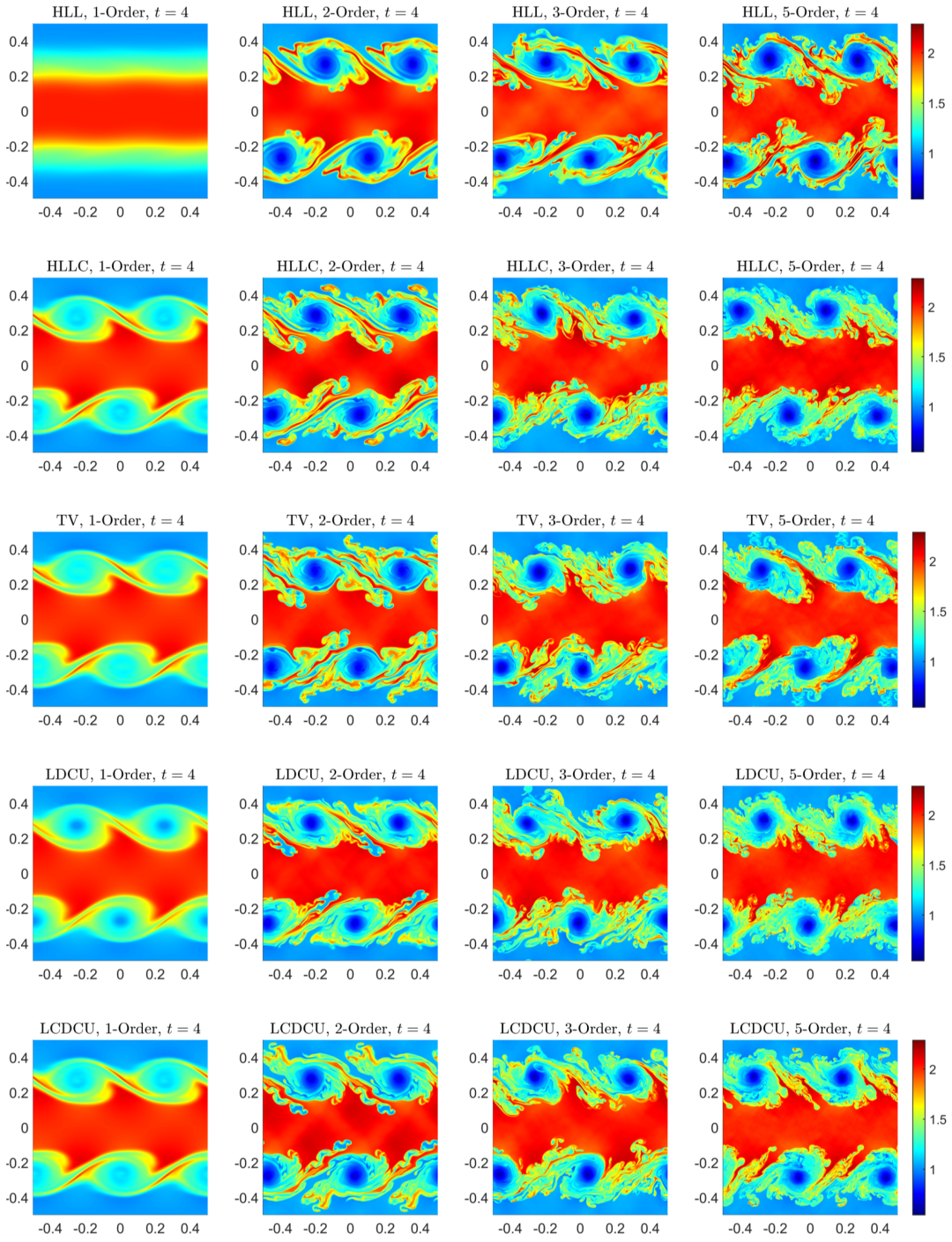


FIGURE 19 | Example 16: Time snapshots of density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes at $t = 4$.

at the final time. This test is particularly effective in revealing odd–even decoupling and other numerical shock instabilities. A shock-stable scheme should maintain a nearly planar shock front, whereas a shock-unstable scheme may produce spurious transverse oscillations.

The obtained numerical results are plotted in Figure 20. As one can see, the HLL schemes maintain essentially planar shock fronts. In contrast, the HLLC, TV, and LCDCU schemes exhibit much more pronounced numerical shock instabilities, while the LDCU schemes produce only weak instabilities compared with the HLL schemes. The third- and fifth-order TV computations encounter a loss of positivity, and their results are therefore omitted from Figure 20.

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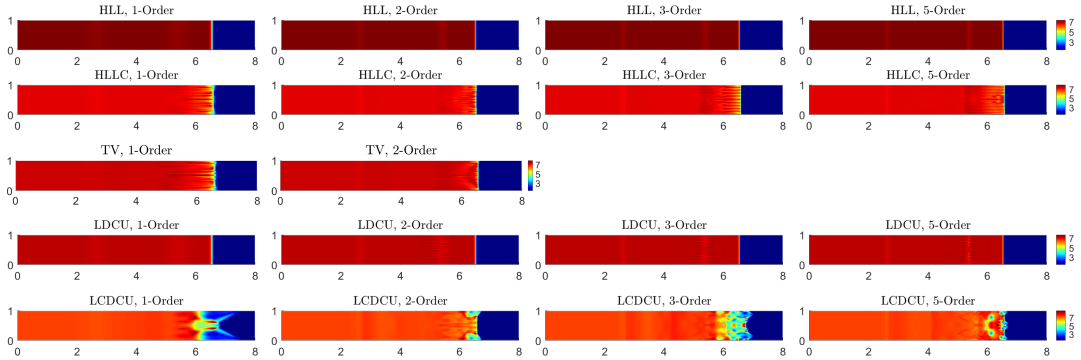


FIGURE 20 | Example 17: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), LDCU (fourth row), and LCDCU (bottom row) schemes, as well as the first- and second-order TV (third row) schemes.

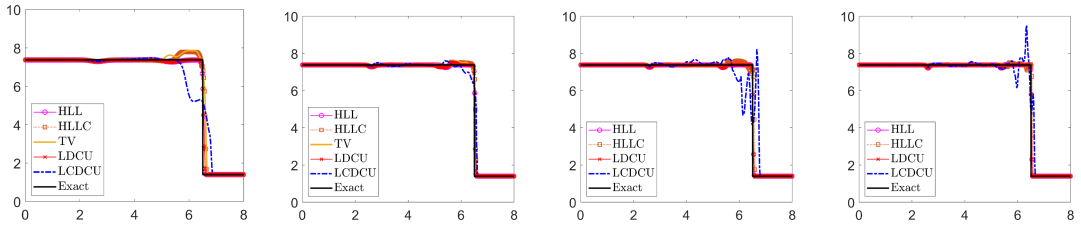


FIGURE 21 | Example 17: Density $[\rho]$ along the centerline $[y = 0.5]$ computed by the studied schemes.

For a quantitative comparison, we also plot the density along the centerline $y = 0.5$. The obtained centerline densities are presented in Figure 21. The results confirm that the HLLC, TV, and LCDCU schemes develop pronounced spurious oscillations, whereas the HLL schemes remain nonoscillatory and the LDCU schemes exhibit only weak deviations from the HLL results.

4.2.10 | Example 18—Shock Diffraction Over a 90° Corner

In this example, we consider a Mach-2 version of the shock diffraction problem over a 90° corner; see, e.g., [90]. The flow region is the L-shaped domain

$$\Omega = ([0, 1] \times [0, 1]) \cup ([1, 2] \times [0, 0.5]).$$

The initial conditions, prescribed in the fluid region Ω, are given by

$$(\rho, u, v, p)(x, y, 0) = \begin{cases} (1,\ 0,\ 0,\ 1), & x < 1.375, \\ \left(\frac{8}{3}, -\frac{\sqrt{35}}{4}, 0, \frac{9}{2}\right), & x > 1.375, \end{cases}$$

which correspond to a left-moving Mach-2 shock initially located at $x = 1.375$. The left and right states are the pre-shock and post-shock states, respectively. Solid-wall boundary conditions are imposed along the channel walls and the boundary of the solid block. At the open left and right boundaries, the corresponding pre-shock and post-shock states are maintained, respectively.

We compute the numerical solutions until the final time $t = 0.4$ using the studied schemes on a uniform Cartesian mesh with $\Delta x = \Delta y = 1/256$. The obtained density distributions are presented in Figure 22. One can see that all of the studied schemes capture the main structure of the diffracted shock wave. Compared with the HLL schemes, the four low-dissipation schemes provide sharper resolution of the complex flow structures generated around the corner. At the same time, the differences among the four low-dissipation schemes are relatively limited.

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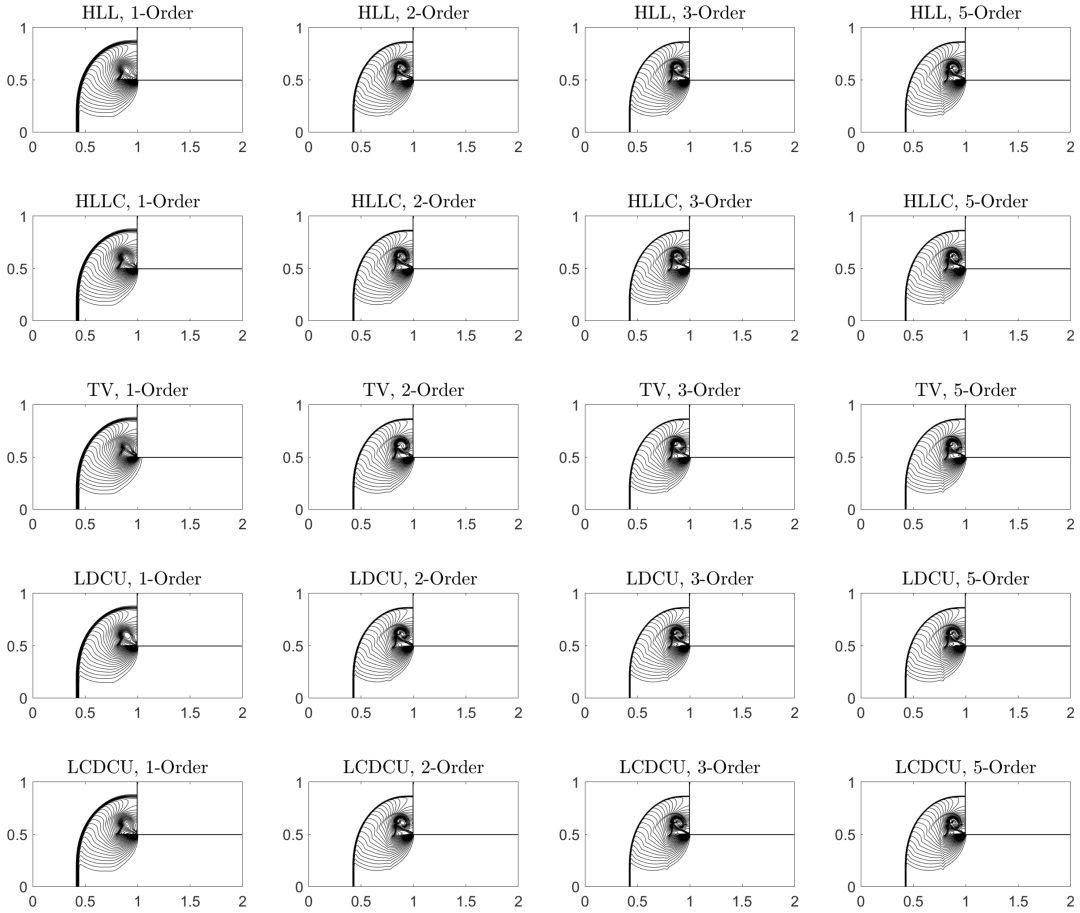


FIGURE 22 | Example 18: Density ρ computed by the first-, second-, third-, and fifth-order HLL (top row), HLLC (second row), TV (third row), LDCU (fourth row), and LCDCU (bottom row) schemes.

4.2.11 | Example 19—Rayleigh-Taylor (RT) Instability

In the last example taken from Reference [91], we investigate the RT instability, which is a physical phenomenon occurring when a layer of heavier fluid is placed on top of a layer of lighter fluid. High-order well-balanced discretizations for the Euler equations with gravitational source terms, including finite-volume schemes and finite-volume/finite-difference WENO formulations, have been developed; see, e.g., [92–94]. The model is governed by the 2-D Euler Equations (3.1) and (3.2) with added gravitational source terms, and the modified system reads as

$$\begin{aligned}
&\rho_t + (\rho u)_x + (\rho v)_y = 0,\\
&(\rho u)_t + (\rho u^2 + p)_x + (\rho u v)_y = 0,\\
&(\rho v)_t + (\rho u v)_x + (\rho v^2 + p)_y = \rho,\\
&E_t + [u(E+p)]_x + [v(E+p)]_y = \rho v.
\end{aligned}$$

We consider the following initial conditions:

$$(\rho(x,y,0), u(x,y,0), v(x,y,0), p(x,y,0)) = \begin{cases} (2, 0, -0.025\ c\cos(8\pi x), 2y+1), & y < 0.5,\\ (1, 0, -0.025\ c\cos(8\pi x), y+1.5), & \text{otherwise,}\end{cases}$$

where $c := \sqrt{\gamma p/\rho}$ is the speed of sound, prescribed in the computational domain $[0, 0.25] \times [0, 1]$ with the solid wall boundary conditions imposed at $x = 0$ and $x = 0.25$, and the following Dirichlet boundary conditions imposed at the top and bottom boundaries:

$$(\rho, u, v, p)|_{y=1} = (1, 0, 0, 2.5), \qquad (\rho, u, v, p)|_{y=0} = (2, 0, 0, 1).$$

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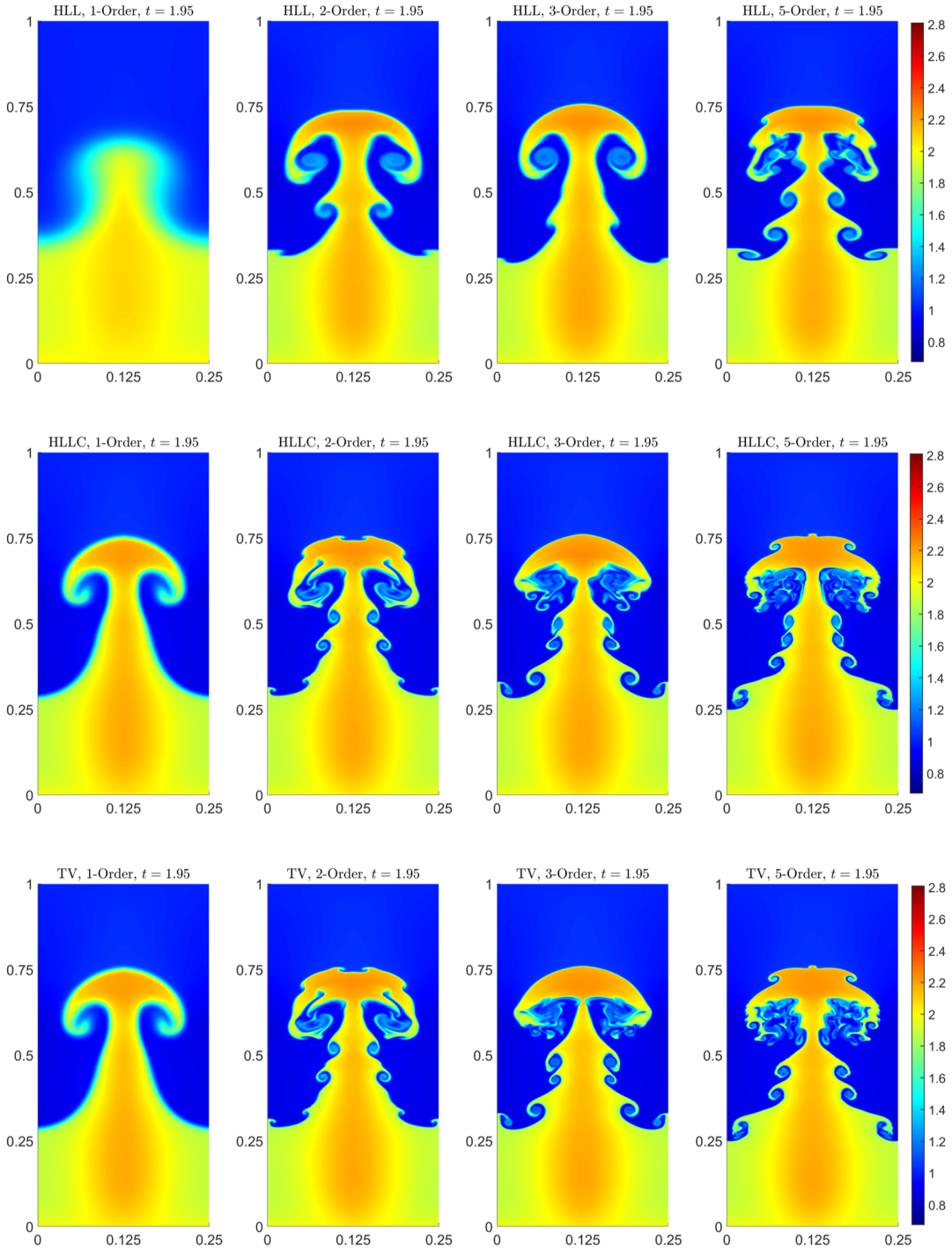


FIGURE 23 | Example 19: Density ρ computed by the first-order (first column), second-order (second column), third-order (third column), and fifth-order (fourth column) HLL (top row), HLLC (second row), and TV (bottom row) schemes at $t = 1.95$.

We compute the numerical solutions until the final time $t = 2.95$ using first-, second-, third-, and fifth-Order schemes on the uniform mesh of 256×1024 cells and present the numerical results at times $t = 1.95$ and 2.95 in Figures 23–26. One can see that there are pronounced differences between the solutions computed by the HLL and four low-dissipation schemes since the structures captured by the low-dissipation schemes are much more complicated, which again demonstrates that the four schemes contain less dissipation than the corresponding HLL counterparts. At the same time, the differences between the studied four low-dissipation schemes are limited, especially at the earlier time; see Figures 23 and 24. At later times, they produce different complex structures because of the unstable nature of the Rayleigh-Taylor instability problem.

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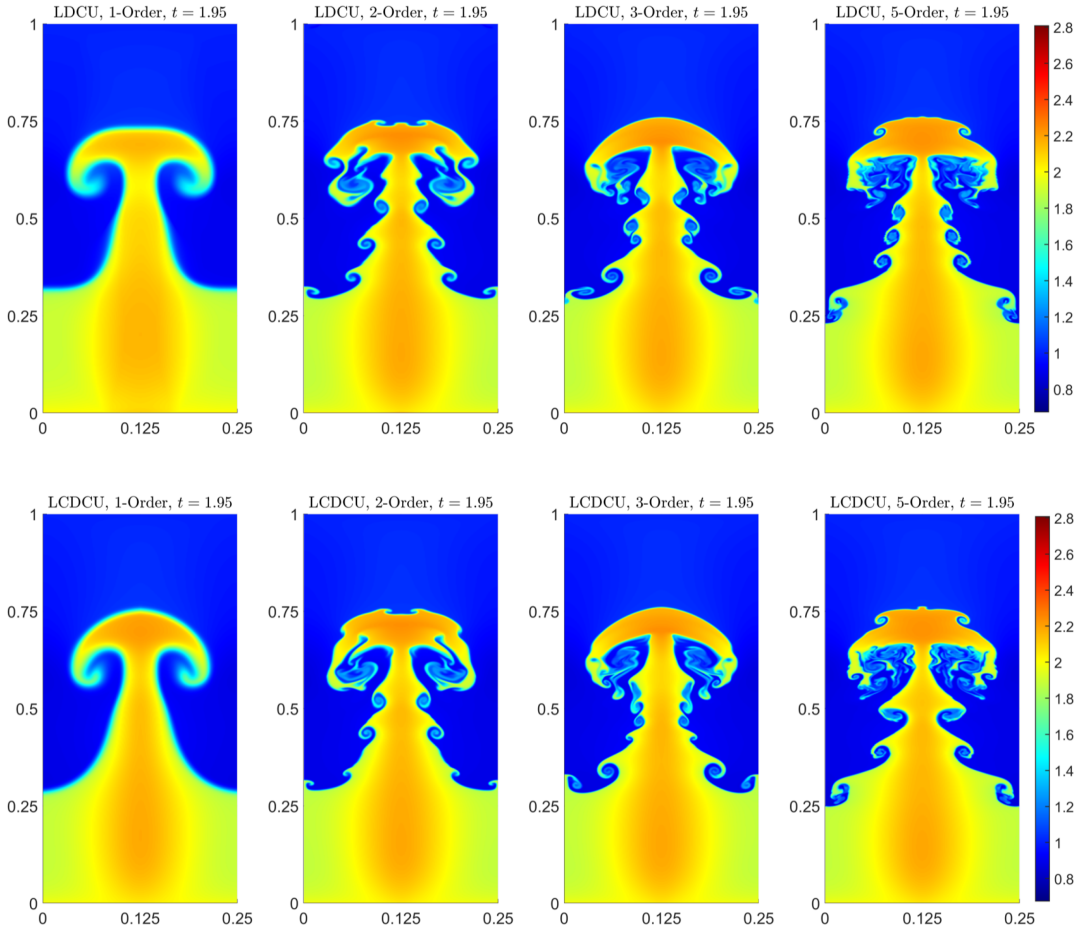


FIGURE 24 | Example 19: Density ρ computed by the first-order (first column), second-order (second column), third-order (third column), and fifth-order (fourth column) LDCU (top row) and LCDCU (bottom row) schemes at $t = 1.95$.

Remark 4.2 (Positivity preservation). Preservation of positive density and pressure is an important robustness property for numerical schemes for the Euler equations. The first-order HLL and HLLC schemes are positivity-preserving under suitable wave-speed estimates and CFL conditions; see [95, 96]. However, these results do not automatically extend to the higher-order versions considered here. To the best of our knowledge, no general positivity-preserving result is available for the unmodified TV and LCDCU schemes used in this study. Moreover, the original LDCU construction may fail to preserve positivity; specially modified PPLDCU schemes have recently been developed to guarantee this property [97]. Since none of our high-order implementations employs a dedicated positivity-preserving limiter, we do not claim a general positivity guarantee for them. Such a guarantee could be enforced by blending the high-order flux with a positivity-preserving first-order flux [98, 99]. The loss of positivity observed for the third- and fifth-order TV schemes in Configuration 3 and Quirk's problem illustrates this issue.

5 | Conclusions

In this paper, we have carried out a systematic comparative study of several low-dissipation numerical schemes for hyperbolic conservation laws, including the HLLC scheme and three recently proposed methods—the TV flux splitting, LDCU, and LCDCU schemes. These schemes have been considered at first-, second-, third-, and fifth-order accuracy levels within finite-volume and finite-difference frameworks. Through a series of one- and two-dimensional numerical experiments for the Euler equations of gas dynamics, we have examined their accuracy, resolution, robustness, and computational efficiency. The numerical results indicate that the four low-dissipation schemes exhibit comparable levels of numerical dissipation, with noticeable differences arising only in selected test cases. The transition from first- to second-order accuracy generally produces the most significant improvement in resolution, while the third- and fifth-order versions further enhance the accuracy and resolution of complex multidimensional flow structures. Compared

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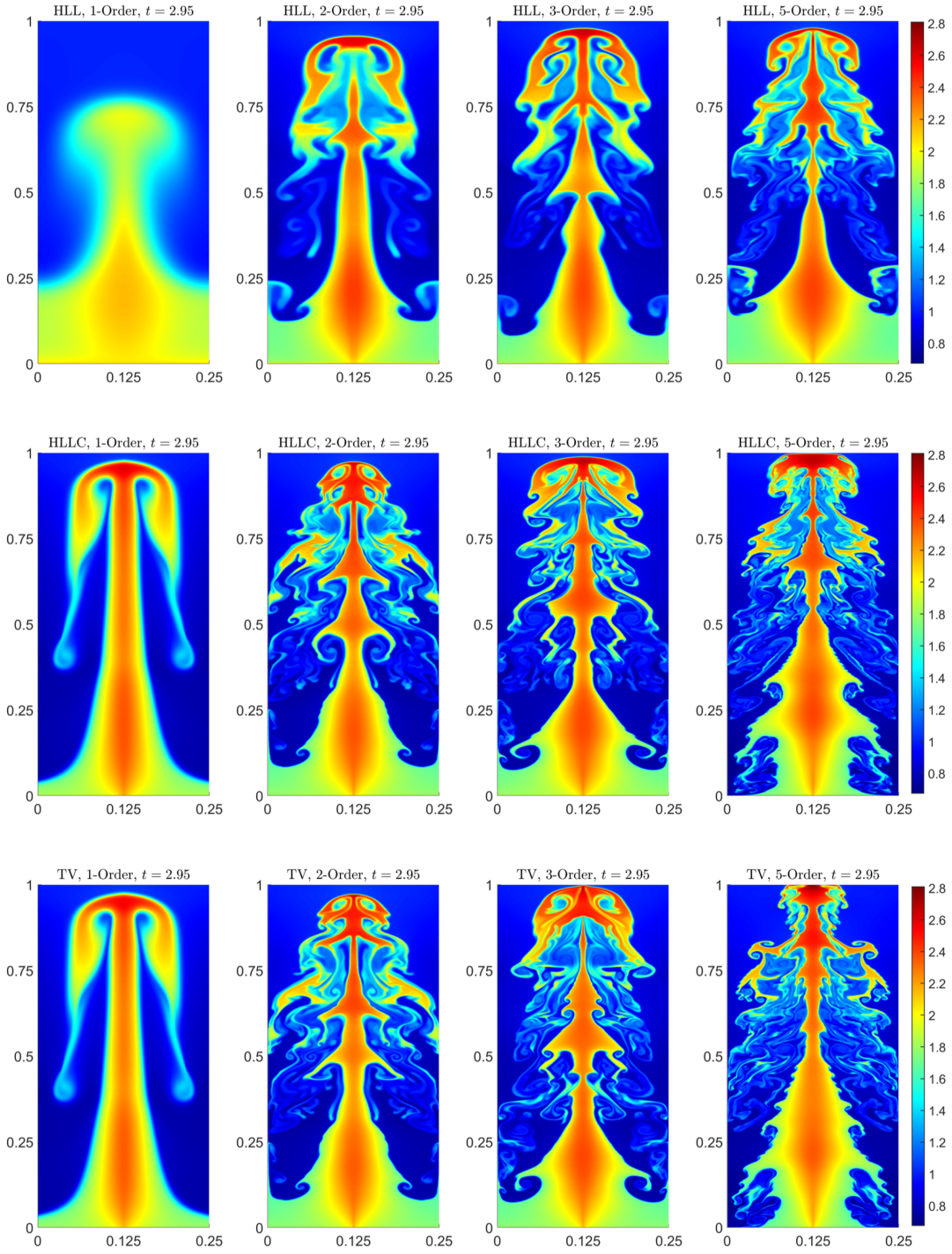


FIGURE 25 | Example 19: Density ρ computed by the first-order (first column), second-order (second column), third-order (third column), and fifth-order (fourth column) HLL (top row), HLLC (second row), and TV (bottom row) schemes at $t = 2.95$.

with their HLL counterparts, all four low-dissipation schemes provide sharper resolution of contact and shear waves. Among them, the TV splitting schemes exhibit slightly higher numerical dissipation in several examples. The newly added shock-instability tests further reveal an important trade-off between resolution and robustness. In Quirk's odd–even decoupling problem, the HLL schemes maintain essentially planar shock fronts, whereas the HLLC, TV, and LCDCU schemes exhibit more pronounced numerical shock instabilities. The LDCU schemes produce only relatively weak deviations from the HLL results. Moreover, the third- and fifth-order TV schemes lose positivity in the two-dimensional Riemann problem — Configuration 3 and Quirk's odd–even decoupling problem. None of the high-order schemes considered in this study is equipped with a dedicated positivity-preserving limiter. Therefore, the absence of positivity failures

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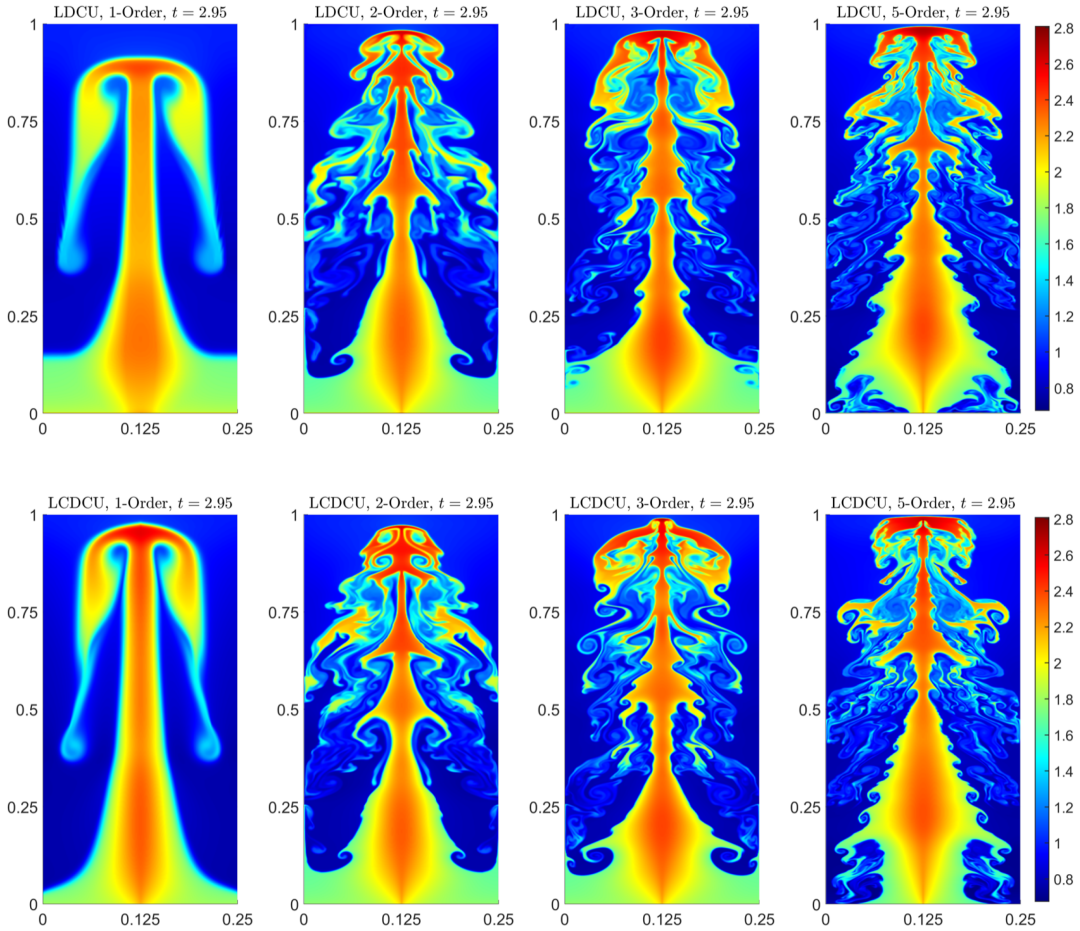


FIGURE 26 | Example 19: Density ρ computed by the first-order (first column), second-order (second column), third-order (third column), and fifth-order (fourth column) LDCU (top row) and LCDCU (bottom row) schemes at $t = 2.95$.

in the other computations should be interpreted as numerical evidence of robustness rather than as a general theoretical guarantee. For problems involving near-vacuum states or extremely strong shocks, suitable positivity-preserving reconstructions or flux-limiting techniques may be required. In terms of computational cost, the LCDCU schemes are slightly more expensive than the HLLC, TV, and LDCU schemes. Nevertheless, the LCDCU framework offers greater flexibility for extension to other hyperbolic systems, since its construction primarily requires the corresponding local eigenstructure. Overall, no single scheme is uniformly superior in all aspects. The HLL schemes offer greater robustness in the presence of strong grid-aligned shocks, whereas the four low-dissipation schemes provide substantially better resolution of contact and shear waves. The comparative findings therefore provide practical guidance for selecting an appropriate numerical scheme according to the required balance among resolution, robustness, positivity, and computational cost.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The data that support the findings of this study and Fortran codes developed by the authors and used to obtain all of the presented numerical results are available from the corresponding author upon reasonable request.

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