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Local Characteristic Decomposition of Equilibrium Variables for Hyperbolic Systems of Balance Laws

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Abstract

This paper is concerned with high-order numerical methods for hyperbolic systems of balance laws. Such methods are typically based on high-order piecewise polynomial reconstructions (interpolations) of the computed discrete quantities. However, such reconstructions (interpolations) may be oscillatory unless the reconstruction (interpolation) procedure is applied to the local characteristic variables via the local characteristic decomposition (LCD). Another challenge in designing accurate and stable high-order schemes is related to enforcing a delicate balance between the fluxes, sources, and nonconservative product terms: a good scheme should be well-balanced (WB) in the sense that it should be capable of exactly preserving certain (physically relevant) steady states. One of the ways to ensure that the reconstruction (interpolation) preserves these steady states is to apply the reconstruction (interpolation) to the equilibrium variables, which are supposed to be constant at the steady states. To achieve this goal and to keep the reconstruction (interpolation) non-oscillatory, we introduce a new LCD of equilibrium variables. We apply the developed technique to the fifth-order Ai-WENO-Z interpolation implemented within the WB A-WENO framework recently introduced in [S. Chu, A. Kurganov, and R. Xin, Beijing J. of Pure and Appl. Math., 2 (2025), pp. 87–113], and illustrate its performance on a variety of numerical examples.

Key words: High-order reconstructions (interpolations); A-WENO schemes; well-balanced schemes; equilibrium variables; local characteristic decomposition.

AMS subject classification: 76M20, 65M06, 35L65, 35L67.

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1 Introduction

This paper is focused on the development of a new local characteristic decomposition (LCD) of equilibrium variables for hyperbolic systems of balance laws, which, in the two-dimensional (2-D) case, read as

$$\boldsymbol{U}_t+\boldsymbol{F}(\boldsymbol{U})_x+\boldsymbol{G}(\boldsymbol{U})_y=B^x(\boldsymbol{U})\boldsymbol{U}_x+B^y(\boldsymbol{U})\boldsymbol{U}_y+\boldsymbol{S}(\boldsymbol{U}). \tag{1.1}$$

Here, x and y are the spatial variables, t is time, $\boldsymbol{U}\in\mathbb{R}^d$ is a vector of unknowns, $\boldsymbol{F}$ and $\boldsymbol{G}$ are the fluxes, $\boldsymbol{S}$ is the source term, and $B^x(\boldsymbol{U})U_x$ and $B^y(\boldsymbol{U})U_y$ are nonconservative product terms.

Development of high-order numerical methods for (1.1) is a challenging task for the following two main reasons. First, solutions of (1.1) may develop discontinuities (even for infinitely smooth initial data) and thus high-order schemes should rely on non-oscillatory high-order reconstructions (interpolations) of the solutions out of the computed discrete quantities. To make these reconstructions (interpolations) non-oscillatory, one typically needs to use nonlinear limiting techniques often applied to the local characteristic variables via the LCD; see, e.g., [8,12,13,17,20–22,26]. The LCD is applied by computing the local values of the Jacobians $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$ and $\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{U}}$, locally switching to the corresponding characteristic variables, performing the reconstruction (interpolation) to these local variables, and then switching back to the original variables $\boldsymbol{U}$.

Second, many (physically relevant) solutions of (1.1) are, in fact, small perturbations of certain steady states, which are supposed to be exactly preserved by good high-order schemes—this is a so-called well-balanced (WB) property. One of the ways to ensure that the high-order reconstruction (interpolation) preserves these steady states is to reconstruct (interpolate) the equilibrium variables instead of the conservative ones. We note that in many cases, the equilibrium variables can be obtained by rewriting the system (1.1) in the following form (see, e.g., [1]):

$$\boldsymbol{U}_t+M^x(\boldsymbol{U})\boldsymbol{E}^x(\boldsymbol{U})_x+M^y(\boldsymbol{U})\boldsymbol{E}^y(\boldsymbol{U})_y=\boldsymbol{0}, \tag{1.2}$$

where

$$\begin{aligned}M^x(\boldsymbol{U})\boldsymbol{E}^x(\boldsymbol{U})_x&=\boldsymbol{F}(\boldsymbol{U})_x-B^x(\boldsymbol{U})\boldsymbol{U}_x-\boldsymbol{S}^x(\boldsymbol{U}),\\ M^y(\boldsymbol{U})\boldsymbol{E}^y(\boldsymbol{U})_y&=\boldsymbol{G}(\boldsymbol{U})_y-B^y(\boldsymbol{U})\boldsymbol{U}_y-\boldsymbol{S}^y(\boldsymbol{U}),\end{aligned} \tag{1.3}$$

and $\boldsymbol{S}^x(\boldsymbol{U})+\boldsymbol{S}^y(\boldsymbol{U})=\boldsymbol{S}(\boldsymbol{U})$. In (1.3), $M^x,M^y\in\mathbb{R}^{d\times d}$ and $\boldsymbol{E}^x(\boldsymbol{U}(x,y))$, $\boldsymbol{E}^y(\boldsymbol{U}(x,y))$ are equilibrium variables, since

$$\boldsymbol{E}^x(\boldsymbol{U}(x,y))=\boldsymbol{E}^x(y)\quad\text{and}\quad\boldsymbol{E}^y(\boldsymbol{U}(x,y))=\boldsymbol{E}^y(x) \tag{1.4}$$

at steady states satisfying $\boldsymbol{E}^x(\boldsymbol{U})_x=\boldsymbol{E}^y(\boldsymbol{U})_y\equiv 0$. Therefore, one may prefer to reconstruct $\boldsymbol{E}^x$ in the x-direction and $\boldsymbol{E}^y$ in the y-direction and then to recalculate the corresponding values of $\boldsymbol{U}$ to ensure that (1.4) is satisfied at the discrete level. However, applying the LCD approach, which is based on the Jacobians $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{U}}$ and $\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{U}}$ and not on the equilibrium variables, may destroy the WB property of the resulting scheme.

In this paper, we introduce a new LCD of equilibrium variables. To this end, we first rewrite the system (1.2) in the following two equivalent (for smooth solution) forms:

$$\begin{aligned}\boldsymbol{E}^x(\boldsymbol{U})_t+C^x(\boldsymbol{U})\boldsymbol{E}^x(\boldsymbol{U})_x+D^x(\boldsymbol{U})\boldsymbol{U}_y&=\tilde{\boldsymbol{I}}^x(\boldsymbol{U}),\\ \boldsymbol{E}^y(\boldsymbol{U})_t+D^y(\boldsymbol{U})\boldsymbol{U}_x+C^y(\boldsymbol{U})\boldsymbol{E}^y(\boldsymbol{U})_y&=\tilde{\boldsymbol{I}}^y(\boldsymbol{U}),\end{aligned} \tag{1.5}$$

where the matrices C^x and C^y and the source terms $\tilde{\boldsymbol{I}}^x$ and $\tilde{\boldsymbol{I}}^y$ are specified in §3. We then compute the matrices C^x and C^y at the grid points and use them to compute the local characteristic equilibrium variables, which are reconstructed (interpolated) to obtain high-order values of $\boldsymbol{E}^x$ and $\boldsymbol{E}^y$, which, in turn, give us the corresponding high-order values of $\boldsymbol{U}$ (solving nonlinear systems of equations may be required). We implement the new LCD technique in the framework of flux globalization based WB alternative weighted essentially non-oscillatory (A-WENO) finite-difference schemes recently introduced in [7]. The local characteristic equilibrium variables are interpolated using the fifth-order affine-invariant WENO-Z (Ai-WENO-Z) interpolations [9,16,23]. The developed A-WENO scheme is applied to four systems of balance laws including the nozzle flow system, the one- and two-layer shallow water equations, and the compressible Euler equations with gravitation. We conduct several numerical experiments to demonstrate the performance of the proposed scheme.

2 Flux Globalization Based WB A-WENO Schemes: An Overview

In this section, we give an overview of the flux globalization based WB A-WENO schemes introduced in [7] for general nonconservative systems.

2.1 1-D Scheme

The one-dimensional (1-D) hyperbolic systems of balance laws

$$\boldsymbol{U}_t+\boldsymbol{F}(\boldsymbol{U})_x=B(\boldsymbol{U})\boldsymbol{U}_x+\boldsymbol{S}(\boldsymbol{U}) \tag{2.1}$$

can be written in an equivalent quasi-conservative form:

$$\boldsymbol{U}_t+\boldsymbol{K}(\boldsymbol{U})_x=\boldsymbol{0},$$

where $\boldsymbol{K}(\boldsymbol{U})$ is a global flux

$$\boldsymbol{K}(\boldsymbol{U})=\boldsymbol{F}(\boldsymbol{U})-\boldsymbol{R}(\boldsymbol{U}),\quad \boldsymbol{R}(\boldsymbol{U})=\int\limits_{\hat{x}}^{x}\Big[B(\boldsymbol{U}(\xi,t))\boldsymbol{U}_\xi(\xi,t)+\boldsymbol{S}(\boldsymbol{U}(\xi,t))\Big]\,\mathrm{d}\xi,$$

and $\hat{x}$ is an arbitrary number.

We first introduce a uniform mesh consisting of the cells $[x_{j-\frac12},x_{j+\frac12}]$ of size $x_{j+\frac12}-x_{j-\frac12}\equiv\Delta x$ centered at $x_j=(x_{j-\frac12}+x_{j+\frac12})/2$, $j=1,\ldots,N$. We assume that at a certain time level t, the approximate solution, realized in terms of its cell centered values $\boldsymbol{U}_j\approx\boldsymbol{U}(x_j,t)$, is available (in the rest of the paper, we will suppress the time-dependence of all of the indexed quantities for the sake of brevity). The solution is then evolved in time by solving the following system of ODEs:

$$\frac{\mathrm{d}\boldsymbol{U}_j}{\mathrm{d}t}=-\frac{\boldsymbol{\mathcal{K}}_{j+\frac12}-\boldsymbol{\mathcal{K}}_{j-\frac12}}{\Delta x}, \tag{2.2}$$

where $\boldsymbol{\mathcal{K}}_{j+\frac12}$ are the fifth-order A-WENO numerical fluxes (see [6,7]):

$$\boldsymbol{\mathcal{K}}_{j+\frac12}=\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac12}-\frac{(\Delta x)^2}{24}(\boldsymbol{K}_{xx})_{j+\frac12}+\frac{7(\Delta x)^4}{5760}(\boldsymbol{K}_{xxxx})_{j+\frac12}.$$

Here, $\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}$ is a finite-volume numerical flux (in the numerical experiments reported in §4, we have used the second-order WB path-conservative central-upwind numerical flux introduced in [14]), and $(\boldsymbol{K}_{xx})_{j+\frac{1}{2}}$ and $(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2}}$ are the high-order correction terms. The numerical fluxes $\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}=\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}(\boldsymbol{U}^\pm_{j+\frac{1}{2}},\widehat{\boldsymbol{U}}^\pm_{j+\frac{1}{2}})$ are computed using the one-sided interpolated values of $\boldsymbol{U}$, and to enforce the WB evolution, one needs to use two copies of those values denoted by $\boldsymbol{U}^\pm_{j+\frac{1}{2}}$ and $\widehat{\boldsymbol{U}}^\pm_{j+\frac{1}{2}}$; see [7, 14] for details. The correction terms $(\boldsymbol{K}_{xx})_{j+\frac{1}{2}}$ and $(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2}}$ are computed using the numerical fluxes $\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}$, which have been already obtained:

$$
\begin{aligned}
(\boldsymbol{K}_{xx})_{j+\frac{1}{2}}&=\frac{1}{12(\Delta x)^2}\Big[-\boldsymbol{\mathcal{K}}^{\rm FV}_{j-\frac{3}{2}}+16\boldsymbol{\mathcal{K}}^{\rm FV}_{j-\frac{1}{2}}-30\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}+16\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{3}{2}}-\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{5}{2}}\Big],\\
(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2}}&=\frac{1}{(\Delta x)^4}\Big[\boldsymbol{\mathcal{K}}^{\rm FV}_{j-\frac{3}{2}}-4\boldsymbol{\mathcal{K}}^{\rm FV}_{j-\frac{1}{2}}+6\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}-4\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{3}{2}}+\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{5}{2}}\Big];
\end{aligned}
$$

see [6] for details. We would like to stress that these correction terms are needed to increase the order of the resulting scheme to the fifth order and that adding these terms typically does not cause oscillations as long as the reconstruction is performed in the local characteristic variables; see, e.g., [3, 12, 17, 24–26].

2.2 2-D Scheme

The 2-D hyperbolic systems of balance laws (1.1) can be similarly written in an equivalent quasi-conservative form:

$$\boldsymbol{U}_t+\boldsymbol{K}(\boldsymbol{U})_x+\boldsymbol{L}(\boldsymbol{U})_y=\boldsymbol{0},$$

where $\boldsymbol{K}(\boldsymbol{U})$ and $\boldsymbol{L}(\boldsymbol{U})$ are the global fluxes

$$
\begin{aligned}
\boldsymbol{K}(\boldsymbol{U})&=\boldsymbol{F}(\boldsymbol{U})-\boldsymbol{R}^x(\boldsymbol{U}),\quad \boldsymbol{R}^x(\boldsymbol{U})=\int\limits_{\hat x}^{x}\Big[B^x(\boldsymbol{U}(\xi,y,t))\boldsymbol{U}_\xi(\xi,y,t)+\boldsymbol{S}^x(\boldsymbol{U}(\xi,y,t))\Big]\,{\rm d}\xi,\\
\boldsymbol{L}(\boldsymbol{U})&=\boldsymbol{G}(\boldsymbol{U})-\boldsymbol{R}^y(\boldsymbol{U}),\quad \boldsymbol{R}^y(\boldsymbol{U})=\int\limits_{\hat y}^{y}\Big[B^y(\boldsymbol{U}(x,\eta,t))\boldsymbol{U}_\eta(x,\eta,t)+\boldsymbol{S}^y(\boldsymbol{U}(x,\eta,t))\Big]\,{\rm d}\eta,
\end{aligned}
$$

and $\hat x$ and $\hat y$ are arbitrary numbers.

Let $[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]\times[y_{k-\frac{1}{2}},y_{k+\frac{1}{2}}]$, $j=1,\cdots,N_x$, $k=1,\cdots,N_y$ be the uniform 2-D cells centered at (x_j,y_k) with $x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}\equiv\Delta x$, $y_{k+\frac{1}{2}}-y_{k-\frac{1}{2}}\equiv\Delta y$, $x_j=(x_{j-\frac{1}{2}}+x_{j+\frac{1}{2}})/2$, and $y_k=(y_{k-\frac{1}{2}}+y_{k+\frac{1}{2}})/2$. We denote by $\boldsymbol{U}_{j,k}\approx\boldsymbol{U}(x_j,y_k,t)$ the computed cell centered values, which are assumed to be available at a certain time level t. The point values $\boldsymbol{U}_{j,k}$ are evolved in time by solving the following system of ODEs:

$$\frac{{\rm d}\boldsymbol{U}_{j,k}}{{\rm d}t}=-\frac{\boldsymbol{\mathcal{K}}_{j+\frac{1}{2},k}-\boldsymbol{\mathcal{K}}_{j-\frac{1}{2},k}}{\Delta x}-\frac{\boldsymbol{\mathcal{L}}_{j,k+\frac{1}{2}}-\boldsymbol{\mathcal{L}}_{j,k-\frac{1}{2}}}{\Delta y},\tag{2.3}$$

where $\boldsymbol{\mathcal{K}}_{j+\frac{1}{2},k}$ and $\boldsymbol{\mathcal{L}}_{j,k+\frac{1}{2}}$ are the fifth-order A-WENO numerical fluxes (see [6, 7]):

$$
\begin{aligned}
\boldsymbol{\mathcal{K}}_{j+\frac{1}{2},k}&=\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2},k}-\frac{(\Delta x)^2}{24}(\boldsymbol{K}_{xx})_{j+\frac{1}{2},k}+\frac{7(\Delta x)^4}{5760}(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2},k},\\
\boldsymbol{\mathcal{L}}_{j,k+\frac{1}{2}}&=\boldsymbol{\mathcal{L}}^{\rm FV}_{j,k+\frac{1}{2}}-\frac{(\Delta y)^2}{24}(\boldsymbol{L}_{yy})_{j,k+\frac{1}{2}}+\frac{7(\Delta y)^4}{5760}(\boldsymbol{L}_{yyyy})_{j,k+\frac{1}{2}}.
\end{aligned}
$$

Here, $\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}$ and $\mathcal{L}^{\rm FV}_{j,k+\frac{1}{2}}$ are finite-volume numerical fluxes and $(\boldsymbol{K}_{xx})_{j+\frac{1}{2},k}$, $(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2},k}$, $(\boldsymbol{L}_{yy})_{j,k+\frac{1}{2}}$, and $(\boldsymbol{L}_{yyyy})_{j,k+\frac{1}{2}}$ are the high-order correction terms. The numerical fluxes $\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}=\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}\big(\boldsymbol{U}^\pm_{j+\frac{1}{2},k}\big)$ and $\mathcal{L}^{\rm FV}_{j,k+\frac{1}{2}}=\mathcal{L}^{\rm FV}_{j,k+\frac{1}{2}}\big(\boldsymbol{U}^\pm_{j,k+\frac{1}{2}}\big)$ are computed using the one-sided interpolated values of $\boldsymbol{U}$, and to enforce the WB evolution, one may need to use the numerical diffusion switch functions; see [15] for an example of such switch function used for the compressible Euler equations with gravitation. The correction terms $(\boldsymbol{K}_{xx})_{j+\frac{1}{2},k}$, $(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2},k}$, $(\boldsymbol{L}_{yy})_{j,k+\frac{1}{2}}$, and $(\boldsymbol{L}_{yyyy})_{j,k+\frac{1}{2}}$ are computed using the numerical fluxes $\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}$ and $\mathcal{L}^{\rm FV}_{j+\frac{1}{2},k}$, which have been already obtained:

$$
\begin{aligned}
(\boldsymbol{K}_{xx})_{j+\frac{1}{2},k}&=\frac{1}{12(\Delta x)^2}\Big[-\mathcal{K}^{\rm FV}_{j-\frac{3}{2},k}+16\mathcal{K}^{\rm FV}_{j-\frac{1}{2},k}-30\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}+16\mathcal{K}^{\rm FV}_{j+\frac{3}{2},k}-\mathcal{K}^{\rm FV}_{j+\frac{5}{2},k}\Big],\\
(\boldsymbol{K}_{xxxx})_{j+\frac{1}{2},k}&=\frac{1}{(\Delta x)^4}\Big[\mathcal{K}^{\rm FV}_{j-\frac{3}{2},k}-4\mathcal{K}^{\rm FV}_{j-\frac{1}{2},k}+6\mathcal{K}^{\rm FV}_{j+\frac{1}{2},k}-4\mathcal{K}^{\rm FV}_{j+\frac{3}{2},k}+\mathcal{K}^{\rm FV}_{j+\frac{5}{2},k}\Big];\\
(\boldsymbol{L}_{yy})_{j,k+\frac{1}{2}}&=\frac{1}{12(\Delta y)^2}\Big[-\mathcal{L}^{\rm FV}_{j,k-\frac{3}{2}}+16\mathcal{L}^{\rm FV}_{j,k-\frac{1}{2}}-30\mathcal{L}^{\rm FV}_{j,k+\frac{1}{2}}+16\mathcal{L}^{\rm FV}_{j,k+\frac{3}{2}}-\mathcal{L}^{\rm FV}_{j,k+\frac{5}{2}}\Big],\\
(\boldsymbol{L}_{yyyy})_{j,k+\frac{1}{2}}&=\frac{1}{(\Delta y)^4}\Big[\mathcal{L}^{\rm FV}_{j,k-\frac{3}{2}}-4\mathcal{L}^{\rm FV}_{j,k-\frac{1}{2}}+6\mathcal{L}^{\rm FV}_{j,k+\frac{1}{2}}-4\mathcal{L}^{\rm FV}_{j,k+\frac{3}{2}}+\mathcal{L}^{\rm FV}_{j,k+\frac{5}{2}}\Big];
\end{aligned}
$$

see [6] for details.

3 Local Characteristic Decomposition of Equilibrium Variables

In this section, we apply the LCD approach introduced in [8,12,13,17,20–22,26] to the Ai-WENO-Z interpolation of the equilibrium variables.

3.1 1-D Case

We first rewrite the system (2.1) in the following equivalent (for smooth solution) form:

$$\boldsymbol{U}_t+M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x=\boldsymbol{0}, \tag{3.1}$$

where

$$M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x=\boldsymbol{F}(\boldsymbol{U})_x-B(\boldsymbol{U})\boldsymbol{U}_x-\boldsymbol{S}(\boldsymbol{U}), \tag{3.2}$$

which vanishes at steady states. In (3.2), $M\in\mathbb{R}^{d\times d}$ and $\boldsymbol{E}$ is the vector of equilibrium variables, which are constant at steady states. We then rewrite (3.1) again as

$$\boldsymbol{E}(\boldsymbol{U})_t+C(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x=\tilde{\boldsymbol{I}}(\boldsymbol{U}). \tag{3.3}$$

We now need to specify C and $\tilde{\boldsymbol{I}}$ in (3.3). To this end, we consider two possible cases.

• If $\boldsymbol{E}(\boldsymbol{U})$ is a local quantity, then we simply multiply (3.2) by $\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{U}}$ and obtain (3.3) with $C(\boldsymbol{U})=\frac{\partial \boldsymbol{E}}{\partial \boldsymbol{U}}M(\boldsymbol{U})$ and $\tilde{\boldsymbol{I}}(\boldsymbol{U})\equiv\boldsymbol{0}$

• If $\boldsymbol{E}(\boldsymbol{U})$ is a global quantity, which can be written as

$$\boldsymbol{E}(\boldsymbol{U})=\boldsymbol{W}(\boldsymbol{U})+\boldsymbol{I}(\boldsymbol{U}),$$

where $\boldsymbol{W}(\boldsymbol{U})$ is a local term and

$$\boldsymbol{I}(\boldsymbol{U})=\int\limits_{\hat{x}}^{x}\boldsymbol{H}(\boldsymbol{U}(\xi,t))\,{\rm d}\xi$$

with $\hat{x}$ being an arbitrary number, then we proceed in a different way. We first multiply (3.1) by $\frac{\partial\boldsymbol{W}}{\partial\boldsymbol{U}}$ and then add $\boldsymbol{I}(\boldsymbol{U})_t$ to both sides of the resulting equation to obtain (3.3) with $C(\boldsymbol{U})=\frac{\partial\boldsymbol{W}}{\partial\boldsymbol{U}}M(\boldsymbol{U})$ and $\tilde{\boldsymbol{I}}(\boldsymbol{U})$ computed as follows:

$$\begin{aligned}\boldsymbol{I}(\boldsymbol{U})_t&=\int\limits_{\hat{x}}^{x}\boldsymbol{H}(\boldsymbol{U}(\xi,t))_t\,{\rm d}\xi=\int\limits_{\hat{x}}^{x}\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{U}}\big(\boldsymbol{U}(\xi,t)\big)\boldsymbol{U}_t(\xi,t)\,{\rm d}\xi\\&=-\int\limits_{\hat{x}}^{x}\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{U}}\big(\boldsymbol{U}(\xi,t)\big)M\big(\boldsymbol{U}(\xi,t)\big)\boldsymbol{E}\big(\boldsymbol{U}(\xi,t)\big)_x\,{\rm d}\xi=:\tilde{\boldsymbol{I}}(\boldsymbol{U}).\end{aligned}$$

In fact, the details on $\tilde{\boldsymbol{I}}$ are not important as this source term does not influence the LCD of equilibrium variables, which is based on the matrix C only.

We then evaluate the matrix C at the grid points to obtain the constant matrices $C_j:=C(\boldsymbol{U}_j)$, which can be diagonalized using the matrices Q_j and Q_j^{-1} to obtain $\Lambda_j=Q_j^{-1}C_jQ_j$, where Λ_j is a diagonal matrix containing the eigenvalues of C_j.

Next, we introduce the local characteristic equilibrium variables in the neighborhood of $x=x_j$:

$$\boldsymbol{\Gamma}_\ell=Q_j^{-1}\boldsymbol{E}_\ell,\quad \ell=j\pm2,j\pm1,j,\tag{3.4}$$

apply the fifth-order Ai-WENO-Z (or any other fifth-order WENO-type) interpolation to evaluate the values $\boldsymbol{\Gamma}^+_{j-\frac12}$ and $\boldsymbol{\Gamma}^-_{j+\frac12}$, and finally obtain

$$\boldsymbol{E}^\pm_{j\mp\frac12}=Q_j\boldsymbol{\Gamma}^\pm_{j\mp\frac12}.\tag{3.5}$$

Equipped with these values, we proceed as in [7, 14] and solve the nonlinear equations (see [2, equations (2.8) and (2.16)]) to recover the values $\boldsymbol{U}^\pm_{j\mp\frac12}$ and $\widehat{\boldsymbol{U}}^\pm_{j\mp\frac12}$ needed to evaluate the numerical fluxes $\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac12}$.

Remark 3.1 *It should be pointed out that the presented LCD-based reconstruction algorithm is different from the one used in, e.g., [5] as the LCD process is now performed at the cell centers* $x=x_j$ *not at the cell interfaces* $x_{j+\frac12}$. *The current approach has three advantages. First, no averaged values of any quantities from cells* j *and* $j+1$ *are required. Second, only five—not six—values of* $\boldsymbol{\Gamma}$ *should be computed for every* j. *Third, we can use only one Ai-WENO-Z interpolant in every cell* C_j *to evaluate* $\boldsymbol{\Gamma}^+_{j-\frac12}$ *and* $\boldsymbol{\Gamma}^-_{j+\frac12}$, *while in the LCD algorithm in [5], one had to use two Ai-WENO-Z interpolants to compute the one-sided values of* $\boldsymbol{\Gamma}^\pm_{j+\frac12}$.

3.2 2-D Case

In the 2-D case, the LCD of the equilibrium variable will be based on the formulation of the 2-D system (1.1) given in (1.5). In fact, to perform the LCD, we will only need to specify C^x and C^y as the terms with D^x, D^y, $\tilde{\boldsymbol{I}}^x$, and $\tilde{\boldsymbol{I}}^y$ do not influence the LCD process. The computation of C^x and C^y is similar to the computation of the matrix C in §3.1.

We begin with C^x and consider two possible cases:

- If $\boldsymbol{E}^x(\boldsymbol{U})$ is a local quantity, then $C^x(\boldsymbol{U}) = \frac{\partial \boldsymbol{E}^x}{\partial \boldsymbol{U}} M^x(\boldsymbol{U})$;
- If $\boldsymbol{E}^x(\boldsymbol{U})$ is a global quantity, which can be written as $\boldsymbol{E}^x(\boldsymbol{U}) = \boldsymbol{W}^x(\boldsymbol{U}) + \boldsymbol{I}^x(\boldsymbol{U})$, where $\boldsymbol{W}^x(\boldsymbol{U})$ is a local term, then $C^x(\boldsymbol{U}) = \frac{\partial \boldsymbol{W}^x}{\partial \boldsymbol{U}} M^x(\boldsymbol{U})$.

Similarly, for C^y, we obtain:

- If $\boldsymbol{E}^y(\boldsymbol{U})$ is a local quantity, then $C^y(\boldsymbol{U}) = \frac{\partial \boldsymbol{E}^y}{\partial \boldsymbol{U}} M^y(\boldsymbol{U})$;
- If $\boldsymbol{E}^y(\boldsymbol{U})$ is a global quantity, which can be written as $\boldsymbol{E}^y(\boldsymbol{U}) = \boldsymbol{W}^y(\boldsymbol{U}) + \boldsymbol{I}^y(\boldsymbol{U})$, where $\boldsymbol{W}^y(\boldsymbol{U})$ is a local term, then $C^y(\boldsymbol{U}) = \frac{\partial \boldsymbol{W}^y}{\partial \boldsymbol{U}} M^y(\boldsymbol{U})$.

We then evaluate the matrices C^x and C^y at the grid points to obtain the constant matrices $C^x_{j,k} := C^x(\boldsymbol{U}_{j,k})$ and $C^y_{j,k} := C^y(\boldsymbol{U}_{j,k})$, which can be diagonalized using the matrices $Q^x_{j,k}$ and $Q^y_{j,k}$ to obtain $\Lambda^x_{j,k} = (Q^x_{j,k})^{-1} C^x_{j,k} Q^x_{j,k}$ and $\Lambda^y_{j,k} = (Q^y_{j,k})^{-1} C^y_{j,k} Q^y_{j,k}$, where $\Lambda^x_{j,k}$ and $\Lambda^y_{j,k}$ are the diagonal matrices containing the eigenvalues of $C^x_{j,k}$ and $C^y_{j,k}$, respectively.

Next, we introduce the local characteristic equilibrium variables in the neighborhood of $(x,y) = (x_j, y_k)$:

$$\boldsymbol{\Gamma}^x_{\ell,k} = (Q^x_{j,k})^{-1} \boldsymbol{E}^x_{\ell,k}, \quad \ell = j \pm 2, j \pm 1, j, \quad \text{and} \quad \boldsymbol{\Gamma}^y_{j,\ell} = (Q^y_{j,k})^{-1} \boldsymbol{E}^y_{j,\ell}, \quad \ell = k \pm 2, k \pm 1, k, \tag{3.6}$$

apply the fifth-order Ai-WENO-Z (or any other fifth-order WENO-type) interpolation in the x- and y- directions to evaluate the values $(\boldsymbol{\Gamma}^x_{j\mp\frac{1}{2},k})^\pm$ and $(\boldsymbol{\Gamma}^y_{j,k\mp\frac{1}{2}})^\pm$, respectively, and finally obtain

$$\big(\boldsymbol{E}^x_{j\mp\frac{1}{2},k}\big)^\pm = Q^x_{j,k} \big(\boldsymbol{\Gamma}^x_{j\mp\frac{1}{2},k}\big)^\pm, \quad \big(\boldsymbol{E}^y_{j,k\mp\frac{1}{2}}\big)^\pm = Q^y_{j,k} \big(\boldsymbol{\Gamma}^y_{j,k\mp\frac{1}{2}}\big)^\pm. \tag{3.7}$$

Next, we show the application of the introduced LCD of equilibrium variables to four particular systems of balance laws.

3.3 Application to the Nozzle Flow System

In this section, we consider the 1-D nozzle flow system, which reads as (2.1) with

$$\boldsymbol{U} = (\sigma\rho, \sigma\rho u)^\top, \quad \boldsymbol{F}(\boldsymbol{U}) = (\sigma\rho u, \sigma\rho u^2 + \sigma p)^\top, \quad B(\boldsymbol{U}) = 0, \quad \boldsymbol{S}(\boldsymbol{U}) = (0, p\sigma_x)^\top,$$

where ρ is the density, u is the velocity, $p(\rho) = \kappa\rho^\gamma$ is the pressure, $\kappa > 0$ and $1 < \gamma < \frac{5}{3}$ are constants, and $\sigma = \sigma(x)$ denotes the cross-section of the nozzle. The studied nozzle flow system admits steady-state solutions satisfying $M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x = \boldsymbol{0}$ with

$$M(\boldsymbol{U}) = \begin{pmatrix} 1 & 0 \\ u & \sigma\rho \end{pmatrix}, \quad \boldsymbol{E}(\boldsymbol{U}) = \begin{pmatrix} q \\ E \end{pmatrix}, \quad q = \sigma\rho u, \quad E = \frac{u^2}{2} + \frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1}.$$

In order to apply the fifth-order Ai-WENO-Z interpolation to the equilibrium variables, we first compute

$$E_j = \frac{u_j^2}{2} + \frac{\kappa\gamma}{\gamma-1}(\rho_j)^{\gamma-1},$$

where $u_j = q_j/(\sigma\rho)_j$, $\rho_j = (\sigma\rho)_j/\sigma_j$, and $\sigma_j = \sigma(x_j)$, and then evaluate the matrices

$$C_j = \begin{pmatrix} u_j & (\sigma\rho)_j \\ \dfrac{\kappa\gamma(\rho_j)^{\gamma-1}}{(\sigma\rho)_j} & u_j \end{pmatrix}, \quad Q_j = \begin{pmatrix} (\sigma\rho)_j & (\sigma\rho)_j \\ -\sqrt{\kappa\gamma}(\rho_j)^{\frac{\gamma-1}{2}} & \sqrt{\kappa\gamma}(\rho_j)^{\frac{\gamma-1}{2}} \end{pmatrix},$$
$$Q_j^{-1} = \frac{1}{2\sqrt{\kappa\gamma}(\sigma\rho)_j(\rho_j)^{\frac{\gamma-1}{2}}} \begin{pmatrix} \sqrt{\kappa\gamma}(\rho_j)^{\frac{\gamma-1}{2}} & -(\sigma\rho)_j \\ \sqrt{\kappa\gamma}(\rho_j)^{\frac{\gamma-1}{2}} & (\sigma\rho)_j \end{pmatrix}.$$

We now implement the LCD of the equilibrium variables followed by the fifth-order Ai-WENO-Z interpolation giving $\boldsymbol{\Gamma}^{\pm}_{j\mp\frac{1}{2}}$ and then $q^{\pm}_{j\mp\frac{1}{2}}$ and $E^{\pm}_{j\mp\frac{1}{2}}$. After that, we apply the same fifth-order Ai-WENO-Z interpolation to obtain the one-sided values of the cross-section of the nozzle $\sigma^{\pm}_{j\mp\frac{1}{2}}$, and then solve the nonlinear equations as described in [7, Equations (3.5) and (3.6)] to obtain $(\sigma\rho)^{\pm}_{j\mp\frac{1}{2}}$ and $(\widehat{\sigma\rho})^{\pm}_{j\mp\frac{1}{2}}$.

3.4 Application to the Saint-Venant System with Manning Friction

In this section, we consider the 1-D Saint-Venant system of shallow water equations with Manning friction, which reads as (2.1) with

$$\boldsymbol{U} = (h, q)^\top, \quad \boldsymbol{F}(\boldsymbol{U}) = \Big(q, hu^2 + \frac{1}{2}gh^2\Big)^\top, \quad B(\boldsymbol{U}) = 0, \quad \boldsymbol{S}(\boldsymbol{U}) = (0, -ghZ_x - ghS_f)^\top,$$

where h is the water depth, u is the velocity, $q = hu$ represents the discharge, $Z(x)$ is a function describing the bottom topography, which can be discontinuous, g is the constant acceleration due to gravity, S_f is the Manning friction term (see, e.g., [19]) given by $S_f = n^2 q|q|h^{-\frac{10}{3}}$. The studied Saint-Venant system admits steady-state solutions satisfying $M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x = \mathbf{0}$ with

$$M(\boldsymbol{U}) = \begin{pmatrix} 1 & 0 \\ u & h \end{pmatrix}, \quad \boldsymbol{E}(\boldsymbol{U}) = \begin{pmatrix} q \\ E \end{pmatrix}, \quad E = \frac{u^2}{2} + g(h+Z) + \int\limits_{\hat{x}}^{x} gS_f\,\mathrm{d}\xi.$$

In order to apply the fifth-order Ai-WENO-Z interpolation to the equilibrium variables, we first compute

$$E_j = \frac{u_j^2}{2} + g(h_j + Z_j) + I_j,$$

where $u_j = q_j/h_j$, $Z_j = Z(x_j)$, and I_j is a fifth-order approximation of the integral $\int_{x_{-\frac{5}{2}}}^{x_j} gS_f\,\mathrm{d}x$, in which we have set $\hat{x} = x_{-\frac{5}{2}}$. The values I_j are computed as follows. First, we evaluate $f_j := g(S_f)_j = gn^2 q_j|q_j|h_j^{-\frac{10}{3}}$ for $j = 1, \ldots, N$ and extend these values to $j = -5, \ldots, 0$ and $j = N+1, \ldots, N+5$ using the prescribed boundary conditions (for h and q) implemented within the ghost cell framework. We then construct the interpolating polynomial for f using the five points $(x_{-\frac{5}{2}},f^+_{-\frac{5}{2}})$, $(x_{-\frac{9}{4}},f_{-\frac{9}{4}})$, (x_{-2},f_{-2}), $(x_{-\frac{7}{4}},f_{-\frac{7}{4}})$, and $(x_{-\frac{3}{2}},f^-_{-\frac{3}{2}})$ and integrate it over the interval $[x_{-\frac{5}{2}},x_{-2}]$ to obtain

$$I_{-2}=\frac{\Delta x}{360}\left[29f^+_{-\frac{5}{2}}+124f_{-\frac{9}{4}}+24f_{-2}+4f_{-\frac{7}{4}}-f^-_{-\frac{3}{2}}\right], \tag{3.8}$$

where $f_{j\pm\frac{1}{4}}$ are evaluated as in [5, §4], but now we use the Ai-WENO-Z interpolant instead of the WENO-Z one implemented in [5].

We then proceed recursively: construct the interpolating polynomials for f using (x_{j-1},f_{j-1}), $(x_{j-\frac{3}{4}},f_{j-\frac{3}{4}})$, $(x_{j-\frac{1}{2}},f_{j-\frac{1}{2}})$, $(x_{j-\frac{1}{4}},f_{j-\frac{1}{4}})$, and (x_j,f_j), integrate them over the corresponding interval $[x_{j-1},x_j]$, and end up with

$$I_j=I_{j-1}+\frac{\Delta x}{90}\left[7f_{j-1}+32f_{j-\frac{3}{4}}+12f_{j-\frac{1}{2}}+32f_{j-\frac{1}{4}}+7f_j\right],\quad j=1,\ldots,N_y+3, \tag{3.9}$$

where, the point values $f_{j-\frac{3}{4}}$ and $f_{j-\frac{1}{4}}$ are computed as in [5, §4], but using the Ai-WENO-Z interpolant and $f_{j-\frac{1}{2}}:=\big(f^-_{j-\frac{1}{2}}+f^+_{j-\frac{1}{2}}\big)/2$.

We then evaluate the matrices

$$C_j=\begin{pmatrix}u_j & h_j\\ g & u_j\end{pmatrix},\quad Q_j=\begin{pmatrix}\sqrt{h_j} & \sqrt{h_j}\\ -\sqrt{g} & \sqrt{g}\end{pmatrix},\quad Q_j^{-1}=\frac{1}{2\sqrt{gh_j}}\begin{pmatrix}\sqrt{g} & -\sqrt{h_j}\\ \sqrt{g} & \sqrt{h_j}\end{pmatrix},$$

and implement the LCD of the equilibrium variables followed by the fifth-order Ai-WENO-Z interpolation giving $\mathbf{\Gamma}^\pm_{j\mp\frac{1}{2}}$ and then $q^\pm_{j\mp\frac{1}{2}}$ and $E^\pm_{j\mp\frac{1}{2}}$. After that, we apply the same fifth-order Ai-WENO-Z interpolation to obtain the one-sided values of the bottom topography $Z^\pm_{j\mp\frac{1}{2}}$, and then solve the nonlinear equations

$$E^\pm_{j+\frac{1}{2}}=\frac{\big(q^\pm_{j+\frac{1}{2}}\big)^2}{2\big(h^\pm_{j+\frac{1}{2}}\big)^2}+g\left(h^\pm_{j+\frac{1}{2}}+Z^\pm_{j+\frac{1}{2}}\right)+I_{j+\frac{1}{2}}, \tag{3.10}$$

and

$$E^\pm_{j+\frac{1}{2}}=\frac{\big(q^\pm_{j+\frac{1}{2}}\big)^2}{2\big(\widehat{h}^\pm_{j+\frac{1}{2}}\big)^2}+g\left(\widehat{h}^\pm_{j+\frac{1}{2}}+Z_{j+\frac{1}{2}}\right)+I_{j+\frac{1}{2}},\quad Z_{j+\frac{1}{2}}=\frac{1}{2}\left(Z^+_{j+\frac{1}{2}}+Z^-_{j+\frac{1}{2}}\right) \tag{3.11}$$

to obtain $h^\pm_{j+\frac{1}{2}}$ and $\widehat{h}^\pm_{j+\frac{1}{2}}$, respectively. Here, the integrals $I_{j+\frac{1}{2}}$ are evaluated recursively by the fifth-order quadrature: we first set $I_{-\frac{5}{2}}=0$ and then compute

$$I_{j+\frac{1}{2}}=I_{j-\frac{1}{2}}+\frac{\Delta x}{90}\left[7f^+_{j-\frac{1}{2}}+32f_{j-\frac{1}{4}}+12f_j+32f_{j+\frac{1}{4}}+7f^-_{j+\frac{1}{2}}\right] \tag{3.12}$$

for $j=-2,\ldots,N+2$, where we have used the quadrature, which is obtained by constructing the interpolating polynomials for f using the five points $(x_{j-\frac{1}{2}},f^+_{j-\frac{1}{2}})$, $(x_{j-\frac{1}{4}},f_{j-\frac{1}{4}})$, (x_j,f_j), $(x_{j+\frac{1}{4}},f_{j+\frac{1}{4}})$, and $(x_{j+\frac{1}{2}},f^-_{j+\frac{1}{2}})$ and integrating them over the corresponding intervals $[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$. As before, the values $f_{j\pm\frac{1}{4}}$ are computed as in [5, §4], but using the Ai-WENO-Z interpolant. Notice that equations (3.10) and (3.11) are cubic and we solve them exactly as described in [2].

3.5 Application to the Two-Layer Shallow Water System

In this section, we consider the 1-D two-layer shallow water system, which reads as (2.1) with

$$\begin{aligned}&\boldsymbol{U}=(h_1,q_1,h_2,q_2)^\top,\\&\boldsymbol{F}(\boldsymbol{U})=\big(q_1,h_1u_1^2+\frac{g}{2}h_1^2,q_2,h_2u_2^2+\frac{g}{2}h_2^2\big)^\top,\\&\boldsymbol{S}(\boldsymbol{U})=(0,-gh_1Z_x,0,-gh_2Z_x)^\top,\end{aligned}\qquad B(\boldsymbol{U})=\begin{pmatrix}0&0&0&0\\0&0&-gh_1&0\\0&0&0&0\\-rgh_2&0&0&0\end{pmatrix}.$$

Here, h_1 and h_2 are the water depths in the upper and lower layers, respectively, u_1 and u_2 are the corresponding velocities, $q_1=h_1u_1$ and $q_2=h_2u_2$ represent the corresponding discharges, $Z(x)$ and g are the same as in §3.4, and $r=\frac{\rho_1}{\rho_2}<1$ is the ratio of the constant densities ρ_1 (upper layer) and ρ_2 (lower layer). The studied two-layer shallow water system admits steady-state solutions satisfying $M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x=\boldsymbol{0}$ with

$$M(\boldsymbol{U})=\begin{pmatrix}1&0&0&0\\u_1&h_1&0&0\\0&0&1&0\\0&0&u_2&h_2\end{pmatrix},\quad \boldsymbol{E}(\boldsymbol{U})=\begin{pmatrix}q_1\\E_1\\q_2\\E_2\end{pmatrix},\qquad \begin{aligned}E_1&:=\frac{q_1^2}{2h_1^2}+g(h_1+h_2+Z),\\E_2&:=\frac{q_2^2}{2h_2^2}+g(rh_1+h_2+Z).\end{aligned}$$

In order to apply the fifth-order Ai-WENO-Z interpolation to the equilibrium variables, we first compute

$$\begin{aligned}(E_1)_j&=\frac{(q_1)_j^2}{2(h_1)_j^2}+g\big[(h_1)_j+(h_2)_j+Z_j\big],\\(E_2)_j&=\frac{(q_2)_j^2}{2(h_2)_j^2}+g\big[r(h_1)_j+(h_2)_j+Z_j\big],\end{aligned}$$

and evaluate the matrices

$$C_j=\begin{pmatrix}(u_1)_j&(h_1)_j&0&0\\g&(u_1)_j&g&0\\0&0&(u_2)_j&(h_2)_j\\rg&0&g&(u_2)_j\end{pmatrix},$$

where $(u_1)_j=(q_1)_j/(h_1)_j$ and $(u_2)_j=(q_2)_j/(h_2)_j$. We then compute the matrices Q_j and Q_j^{-1} numerically and implement the LCD of the equilibrium variables followed by the fifth-order Ai-WENO-Z interpolation giving $\boldsymbol{\Gamma}^\pm_{j\mp\frac12}$ and then $\boldsymbol{E}^\pm_{j\mp\frac12}=\big((q_1)^\pm_{j\mp\frac12},(E_1)^\pm_{j\mp\frac12},(q_2)^\pm_{j\mp\frac12},(E_2)^\pm_{j\mp\frac12}\big)^\top$. After that, we apply the same fifth-order Ai-WENO-Z interpolation to obtain the one-sided values of the bottom topography $Z^\pm_{j\mp\frac12}$, and then solve the nonlinear equations as described in [7, Equations (3.11)–(3.14)] to obtain $(h_i)^\pm_{j+\frac12}$ and $(\widehat{h}_i)^\pm_{j+\frac12}$, $i=1,2$.

3.6 Application to the 1-D Euler Equations with Gravitation

In this section, we consider the 1-D compressible Euler equations with gravitation, which can be written as (2.1) with

$$\boldsymbol{U}=(\rho,m,\mathcal{E})^\top,\quad \boldsymbol{F}(\boldsymbol{U})=\big(m,\rho u^2+p,u(\mathcal{E}+p)\big)^\top,\quad B(\boldsymbol{U})\equiv0,\quad \boldsymbol{S}(\boldsymbol{U})=(0,-\rho\phi_x,0)^\top,$$

where ρ is the density, u is the velocity, $m := \rho u$ is the momentum, $\mathcal{E} := E + \rho\phi$, E is the total energy, p is the pressure, and $\phi(x)$ is the time-independent gravitational potential. The system is completed with an equation of state (EOS), which, in the case of ideal gas, reads as $E = \frac{p}{\gamma-1} + \frac{1}{2}\rho u^2$, where γ is a specific heat ratio. The studied 1-D Euler equations with gravitation admit steady-state solutions satisfying $M(\boldsymbol{U})\boldsymbol{E}(\boldsymbol{U})_x = \mathbf{0}$ with

$$M(\boldsymbol{U}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ L & 0 & m \end{pmatrix}, \quad \boldsymbol{E}(\boldsymbol{U}) = \begin{pmatrix} m \\ K \\ L \end{pmatrix}, \quad K := \rho u^2 + p + \int\limits_{\hat{x}}^{x} \rho\phi_x \,\mathrm{d}\xi, \quad L := \frac{\mathcal{E} + p}{\rho}.$$

In order to apply the fifth-order Ai-WENO-Z interpolation to the equilibrium variables, we first compute

$$K_j = \rho_j u_j^2 + p_j + I_j, \quad L_j = \frac{\mathcal{E}_j + p_j}{\rho_j}$$

where I_j is a fifth-order approximation of the integral $\int_{x_{-\frac{5}{2}}}^{x_j} \rho\phi_x \,\mathrm{d}x$, in which we have set $\hat{x} = x_{-\frac{5}{2}}$. The values I_j are computed as follows. First, we evaluate $f_j := \rho_j\phi_x(x_j)$ for $j = 1, \ldots, N$ and extend these values to $j = -5, \ldots, 0$ and $j = N+1, \ldots, N+5$ using the prescribed boundary conditions implemented within the ghost cell framework. We then compute the values I_j, $j = -2, \ldots, N+3$ using (3.8)–(3.9).

We then evaluate the matrices

$$C_j = \begin{pmatrix} 0 & 1 & 0 \\ (\gamma-2)u_j^2 + c_j^2 & (3-\gamma)u_j & (\gamma-1)m_j \\ \dfrac{(\gamma-1)u_j^2 + c_j^2}{\rho_j} & \dfrac{(1-\gamma)u_j}{\rho_j} & \gamma u_j \end{pmatrix}, \quad Q_j = \begin{pmatrix} \dfrac{(1-\gamma)m_j}{c_j^2} & -\dfrac{\rho_j}{c_j} & \dfrac{\rho_j}{c_j} \\ \dfrac{(1-\gamma)m_j^2}{c_j^2\rho_j} & \rho_j - \dfrac{m_j}{c_j} & \rho_j + \dfrac{m_j}{c_j} \\ 1 & 1 & 1 \end{pmatrix},$$

$$Q_j^{-1} = \begin{pmatrix} \dfrac{u_j}{\rho_j} & -\dfrac{1}{\rho_j} & 1 \\ \dfrac{(1-\gamma)u_j^2}{2c_j\rho_j} - \dfrac{1}{2\rho_j}(u_j + c_j) & \dfrac{(\gamma-1)u_j}{2c_j\rho_j} + \dfrac{1}{2\rho_j} & \dfrac{(1-\gamma)u_j}{2c_j} \\ \dfrac{(\gamma-1)u_j^2}{2c_j\rho_j} - \dfrac{1}{2\rho_j}(u_j - c_j) & \dfrac{(1-\gamma)u_j}{2c_j\rho_j} + \dfrac{1}{2\rho_j} & \dfrac{(\gamma-1)u_j}{2c_j} \end{pmatrix},$$

where $c_j := \sqrt{\gamma p_j/\rho_j}$, and implement the LCD of the equilibrium variables followed by the fifth-order Ai-WENO-Z interpolation giving $\boldsymbol{\Gamma}^{\pm}_{j\mp\frac{1}{2}}$ and then $m^{\pm}_{j\mp\frac{1}{2}}$, $K^{\pm}_{j\mp\frac{1}{2}}$, and $L^{\pm}_{j\mp\frac{1}{2}}$. After that, we solve the quadratic equations

$$(\gamma-1)\big(L^{\pm}_{j+\frac{1}{2}} - \phi_x(x_{j+\frac{1}{2}})\big)\big(\rho^{\pm}_{j+\frac{1}{2}}\big)^2 - \gamma\big(K^{\pm}_{j+\frac{1}{2}} - I_{j+\frac{1}{2}}\big)\rho^{\pm}_{j+\frac{1}{2}} + \frac{(\gamma+1)\big(m^{\pm}_{j+\frac{1}{2}}\big)^2}{2} = 0 \tag{3.13}$$

using the method described in [15, Appendix B.1] to obtain $\rho^{\pm}_{j\mp\frac{1}{2}}$ and then $\mathcal{E}^{\pm}_{j\mp\frac{1}{2}}$. In (3.13), the integrals $I_{j+\frac{1}{2}}$ are evaluated recursively using (3.12). For the numerical flux $\boldsymbol{\mathcal{K}}^{\rm FV}_{j+\frac{1}{2}}$, we use the one given in [15, (2.2)].

3.7 Application to the 2-D Euler Equations with Gravitation

In this section, we consider the 2-D Euler equations with gravitation, which reads as (1.1) with

$$\begin{aligned}&\boldsymbol{U}=(\rho,m,n,\mathcal{E})^\top,\ \boldsymbol{F}(\boldsymbol{U})=\Big(m,\rho u^2+p,\frac{mn}{\rho},u(\mathcal{E}+p)\Big)^\top,\ \boldsymbol{G}(\boldsymbol{U})=\Big(n,\frac{mn}{\rho},\rho v^2+p,v(\mathcal{E}+p)\Big)^\top,\\&B^x(\boldsymbol{U})=B^y(\boldsymbol{U})\equiv 0,\quad \boldsymbol{S}(\boldsymbol{U})=(0,-\rho\phi_x,-\rho\phi_y,0)^\top,\end{aligned}$$

where v is the y-velocity, $n:=\rho v$ is the y-momentum, and the rest of the notation is the same as in the 1-D case. The system is completed with the EOS for an ideal gas: $E=\frac{p}{\gamma-1}+\frac{1}{2}\rho(u^2+v^2)$.

The studied 2-D Euler equations with gravitation admit steady-state solutions satisfying $M^x(\boldsymbol{U})\boldsymbol{E}^x(\boldsymbol{U})_x=\boldsymbol{0}$ and $M^y(\boldsymbol{U})\boldsymbol{E}^y(\boldsymbol{U})_y=\boldsymbol{0}$ with

$$\begin{aligned}&M^x(\boldsymbol{U})=\begin{pmatrix}1&0&0&0\\0&0&1&0\\v+(\gamma+1)u\psi^x&u+(1-\gamma)v\psi^x&-\gamma\psi^x&(\gamma-1)\rho\psi^x\\L&0&0&m\end{pmatrix},\\&M^y(\boldsymbol{U})=\begin{pmatrix}0&1&0&0\\v+(1-\gamma)u\psi^y&u+(\gamma+1)v\psi^y&-\gamma\psi^y&(\gamma-1)\rho\psi^y\\0&0&1&0\\0&L&0&n\end{pmatrix},\\&\boldsymbol{E}^x(\boldsymbol{U})=(m,n,K^x,L)^\top,\quad \boldsymbol{E}^y(\boldsymbol{U})=(m,n,K^y,L)^\top,\end{aligned}$$

where

$$\begin{aligned}&K^x:=\rho u^2+p+\int\limits_{\hat x}^{x}\rho\phi_x\,{\rm d}\xi,\quad K^y:=\rho v^2+p+\int\limits_{\hat y}^{y}\rho\phi_y\,{\rm d}\eta,\quad L:=\frac{\mathcal{E}+p}{\rho},\\&\psi^x:=\frac{2mn}{(\gamma-1)(2\rho^2L+n^2)-(\gamma+1)m^2},\quad \psi^y:=\frac{2mn}{(\gamma-1)(2\rho^2L+m^2)-(\gamma+1)n^2}.\end{aligned}$$

In order to apply the fifth-order Ai-WENO-Z interpolation to the equilibrium variables, we first compute

$$K^x_{j,k}=\rho_{j,k}u^2_{j,k}+p_{j,k}+I^x_{j,k},\quad K^y_{j,k}=\rho_{j,k}v^2_{j,k}+p_{j,k}+I^y_{j,k},\quad L_{j,k}=\frac{\mathcal{E}_{j,k}+p_{j,k}}{\rho_{j,k}},$$

where $u_{j,k}=m_{j,k}/\rho_{j,k}$, $v_{j,k}=n_{j,k}/\rho_{j,k}$, and $I^x_{j,k}$ and $I^y_{j,k}$ are fifth-order approximations of the integral $\int_{x_{-\frac52}}^{x_j}\rho\phi_x\,{\rm d}x$ and $\int_{y_{-\frac52}}^{y_k}\rho\phi_y\,{\rm d}y$, respectively. The values $I^x_{j,k}$ are computed as follows. For each $k=1,\dots,N_y$, we evaluate $f_{j,k}:=\rho_{j,k}\phi_x(x_j,y_k)$ for $j=1,\dots,N_x$, and extend these values to $j=-5,\dots,0$ and $j=N+1,\dots,N+5$ using the prescribed boundary conditions implemented within the ghost cell framework. We then compute $I^x_{j,k}$, $j=-2,\dots,N_x+3$ using (3.8)–(3.9). The values of $I^y_{j,k}$, $j=1,\dots,N_x$, $k=-2,\dots,N_y+3$ can be computed similarly.

We then take the local parts of $\boldsymbol{E}^x(\boldsymbol{U})$ and $\boldsymbol{E}^y(\boldsymbol{U})$, which are $W^x(\boldsymbol{U})=(m,n,\rho u^2+p,L)$ and $W^y(\boldsymbol{U})=(m,n,\rho v^2+p,L)$, and evaluate their Jacobians

$$\frac{\partial W^x}{\partial U}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ (1-\gamma)\phi+\dfrac{\gamma-3}{2}u^2+\dfrac{\gamma-1}{2}v^2 & (3-\gamma)u & (1-\gamma)v & \gamma-1\\ (\gamma-1)\dfrac{u^2+v^2}{\rho}-\dfrac{\gamma\mathcal{E}}{\rho^2} & (1-\gamma)\dfrac{u}{\rho} & (1-\gamma)\frac{v}{\rho} & \dfrac{\gamma}{\rho}\end{pmatrix},$$

$$\frac{\partial W^y}{\partial U}(\boldsymbol{U})=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ (1-\gamma)\phi+\dfrac{\gamma-1}{2}u^2+\dfrac{\gamma-3}{2}v^2 & (1-\gamma)u & (3-\gamma)v & \gamma-1\\ (\gamma-1)\dfrac{u^2+v^2}{\rho}-\dfrac{\gamma\mathcal{E}}{\rho^2} & (1-\gamma)\dfrac{u}{\rho} & (1-\gamma)\frac{v}{\rho} & \dfrac{\gamma}{\rho}\end{pmatrix}.$$

Notice that computing the matrices $C^x_{j,k}=\frac{\partial W^x}{\partial U}(\boldsymbol{U}_{j,k})M^x(\boldsymbol{U}_{j,k})$ and $C^y_{j,k}=\frac{\partial W^y}{\partial U}(\boldsymbol{U}_{j,k})M^y(\boldsymbol{U}_{j,k})$ analytically is quite cumbersome and their evaluation will be computationally expensive. We therefore compute $C^x_{j,k}$ and $C^y_{j,k}$ together with the matrices $Q^x_{j,k}$, $(Q^x_{j,k})^{-1}$, $Q^y_{j,k}$, and $(Q^y_{j,k})^{-1}$ numerically. We then implement the LCD of the equilibrium variables followed by the fifth-order Ai-WENO-Z interpolation in the x- and y-directions separately. In the x-direction, this gives $\boldsymbol{\Gamma}^\pm_{j\mp\frac12,k}$ and then $m^\pm_{j\mp\frac12,k}$, $n^\pm_{j\mp\frac12,k}$, $(K^x)^\pm_{j\mp\frac12,k}$, and $L^\pm_{j\mp\frac12,k}$. After that, we solve the quadratic equations

$$\begin{aligned}(\gamma-1)\big(L^\pm_{j+\frac12,k}-\phi(x_{j+\frac12},y)\big)\big(\rho^\pm_{j+\frac12,k}\big)^2-\gamma\big((K^x)^\pm_{j+\frac12,k}-I^x_{j+\frac12,k}\big)\rho^\pm_{j+\frac12,k}&\\ +\frac{(\gamma+1)\big(m^\pm_{j+\frac12,k}\big)^2}{2}-\frac{(\gamma-1)\big(n^\pm_{j+\frac12,k}\big)^2}{2}&=0\end{aligned}\tag{3.14}$$

using the method described in [15, Appendix B.2] to obtain $\rho^\pm_{j\mp\frac12,k}$ and then $\mathcal{E}^\pm_{j\mp\frac12,k}$. In (3.14), the integrals $I^x_{j+\frac12,k}$ are evaluated recursively using (3.12). The point values $m^\pm_{j,k\mp\frac12}$, $n^\pm_{j,k\mp\frac12}$, $(K^y)^\pm_{j,k\mp\frac12}$, $L^\pm_{j,k\mp\frac12}$, $\rho^\pm_{j,k\mp\frac12}$, and $\mathcal{E}^\pm_{j,k\mp\frac12}$ can be computed using a similar interpolation procedure carried out in the y-direction. Finally, we use the numerical fluxes described in [15, (2.7)] for $\boldsymbol{\mathcal{K}}_{j+\frac12,k}$ and $\boldsymbol{\mathcal{L}}_{j,k+\frac12}$.

4 Numerical Examples

In this section, we test the proposed fifth-order WB A-WENO scheme based on the LCD of equilibrium variables on several numerical examples for the nozzle flow system, one- and two-layer shallow water equations, and compressible Euler equations with gravitation. For the sake of brevity, this scheme will be referred to as Scheme 1 and its performance will be compared with the following two schemes:

• Scheme 2: The WB A-WENO scheme from [7], in which the Ai-WENO-Z interpolation is applied to the equilibrium variables $\boldsymbol{E}$ without any LCD;

• Scheme 3: The A-WENO scheme from [5], which can only preserve the simplest "lake-at-rest" steady states, but use the LCD applied to the conservative variables $\boldsymbol{U}$ to reduce the spurious oscillations.

In all of the examples, we have solved the ODE systems (2.2) and (2.3) and using the three-stage third-order strong stability preserving (SSP) Runge-Kutta solver (see, e.g., [10,11]) with the time-step restricted by the CFL number 0.5.

4.1 Nozzle Flow System

Example 1—Flow in Continuous Divergent Nozzle

In the first example taken from [7,14], we consider the divergent nozzle described using the smooth cross-section

$$\sigma(x)=0.976+0.748\tanh(0.8x-4).$$

We first take the steady states with $q_{\rm eq}(x)\equiv 8$, $E_{\rm eq}(x)\equiv 21.9230562619897$ and compute the discrete values of $\rho_{\rm eq}(x)$ by solving the corresponding nonlinear equations; see [7, 14]. We then obtain $u_{\rm eq}(x)=q_{\rm eq}(x)/(\sigma(x)\rho_{\rm eq}(x))$.

Equipped with these steady states, we add a small perturbation to the density field and consider the initial data

$$\rho(x,0)=\rho_{\rm eq}(x)+\begin{cases}10^{-2}, & x\in[0.5,1.5],\\ 0, & \text{otherwise},\end{cases}\qquad q(x,0)=\sigma(x)\rho(x,0)u_{\rm eq}(x),$$

which are prescribed in the computational domain $[0,10]$ subject to the free boundary conditions.

We compute the numerical solutions until the final time $t=0.8$ by Schemes 1–3 on a uniform mesh with $\Delta x=1/20$ and plot the differences $\rho(x,0.8)-\rho_{\rm eq}(x)$ and $q(x,0.8)-q_{\rm eq}(x)$ in Figure 4.1. As one can see, Scheme 1 clearly outperforms Schemes 2 and 3 as there are no oscillations in the results computed by Scheme 1. We also stress that in this example, Scheme 3, which is not WB, produces the largest oscillations as we capture small perturbations of the steady state.

Example 2—Flow in Continuous Convergent Nozzle

In the second example also taken from [7, 14], we consider the convergent nozzle described using the smooth cross-sections

$$\sigma(x)=0.976-0.748\tanh(0.8x-4).$$

We first take the steady states with $q_{\rm eq}(x)\equiv 8$, $E_{\rm eq}(x)\equiv 58.3367745090349$, compute the discrete values of $\rho_{\rm eq}(x)$ by solving the corresponding nonlinear equations, and then obtain $u_{\rm eq}(x)=q_{\rm eq}(x)/(\sigma(x)\rho_{\rm eq}(x))$; see [7, 14]. We then consider the initial data containing a substantially larger perturbation than the one studied in Example 1:

$$\rho(x,0)=\rho_{\rm eq}(x)+\begin{cases}0.3, & x\in[0.5,1.5],\\ 0, & \text{otherwise},\end{cases}\qquad q(x,0)=\sigma(x)\rho(x,0)u_{\rm eq}(x).$$

As in Example 1, the computational domain is $[0,10]$ and the free boundary conditions are imposed.

We compute the numerical solutions until the final time $t=0.5$ by Schemes 1–3 on a uniform mesh with $\Delta x=1/20$ and plot the differences $\rho(x,0.5)-\rho_{\rm eq}(x)$ and $q(x,0.5)-q_{\rm eq}(x)$ in Figure

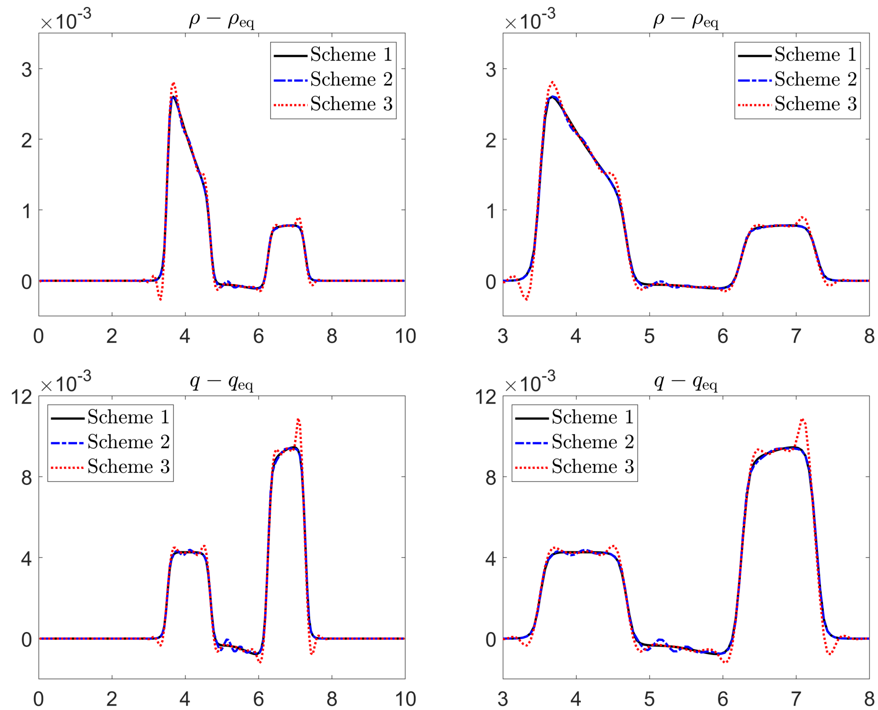

Figure 4.1: Example 1: The differences $\rho(x, 0.8) - \rho_{eq}(x)$ (top row) and $q(x, 0.8) - q_{eq}(x)$ (bottom row) computed by Schemes 1–3, and zoom at $x \in [3, 8]$ (right column).

4.2. As one can see, Scheme 3 does not produce visible oscillations as the magnitude of the perturbation is apparently larger than the size of the truncation errors. On the contrary, Scheme 2, which does not use any LCD, now produces larger oscillations than in Example 1, where the size of perturbation was much smaller.

4.2 Saint-Venant System with Manning Friction

Example 3—Riemann Problem ($n = 0.4$)

In this example, we test the performance of Schemes 1–3 on a Riemann problem with the following initial data (prescribed in the computational domain $[-0.1, 0.3]$ subject to the free boundary conditions):

$$h(x, 0) = \begin{cases} 1, & x < 0, \\ 0.8, & x > 0, \end{cases} \qquad u(x, 0) = \begin{cases} 2, & x < 0, \\ 4 & x > 0, \end{cases}$$

and the bottom topography also containing a jump at $x = 0$:

$$Z(x) = \begin{cases} 1, & x < 0, \\ 1.9 & x > 0. \end{cases}$$

We compute the numerical solutions by the three studied schemes until the final time $t = 0.03$ on a uniform mesh with $\Delta x = 1/250$. The obtained results (h, q, and E) are shown in Figure 4.3, where one can observe that the solutions computed by both Schemes 2 and 3 are oscillatory, whereas Scheme 1 solution is oscillation-free. In order to further study the performance of Schemes

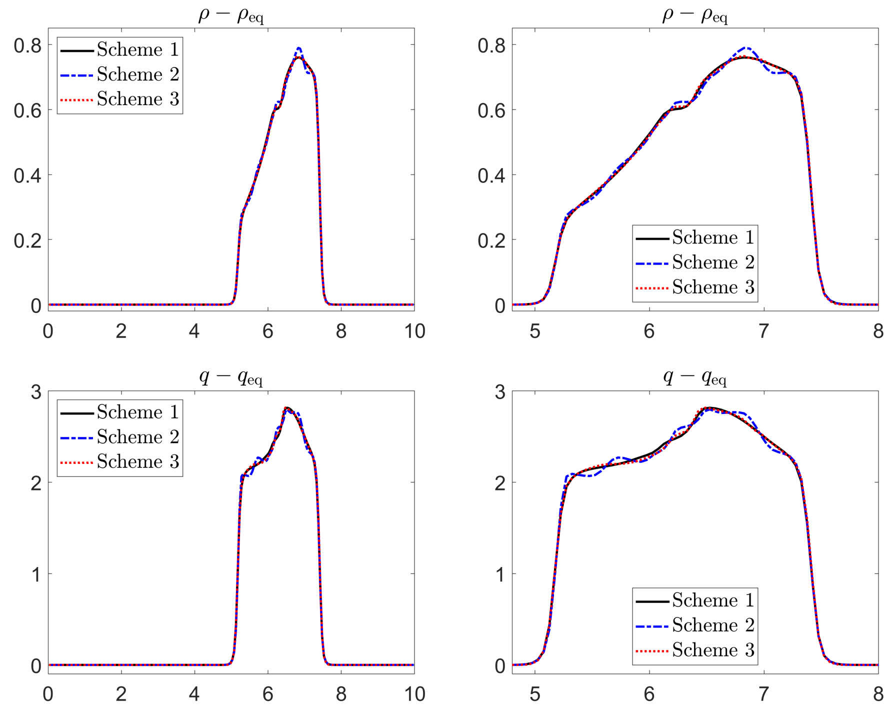

Figure 4.2: Example 2: The differences $\rho(x, 0.5) - \rho_{eq}(x)$ (top row) and $q(x, 0.5) - q_{eq}(x)$ (bottom row) computed by Schemes 1–3, and zoom at $x \in [4.8, 8]$ (right column).

1–3, we refine the mesh to $\Delta x = 1/2500$ and compare the high-resolution solutions of the three studied schemes; see Figure 4.4. One can observe that the high-frequency oscillations produced by Scheme 3 on a coarse mesh, were almost suppressed when the mesh was refined, whereas Scheme 2, which does not employ any LCD, still produces oscillations, which can be clearly seen in the zoom views shown in the right column.

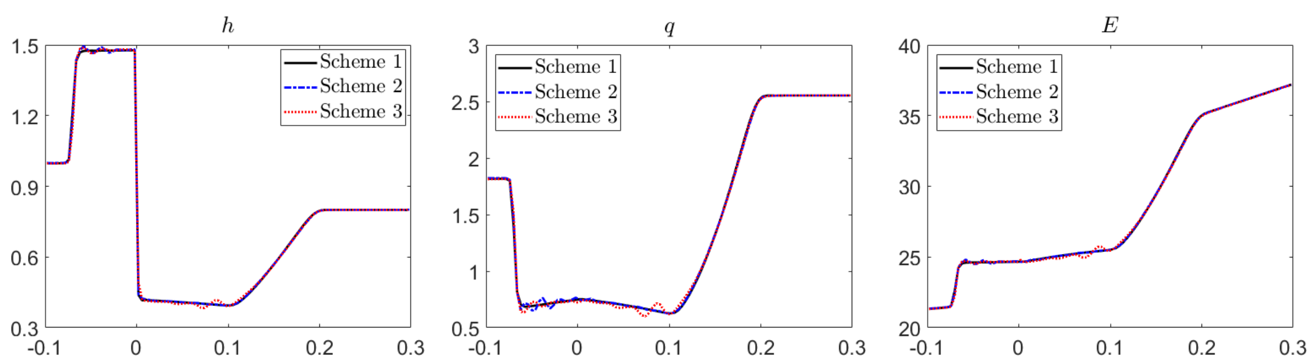

Figure 4.3: Example 3: Water depth h, discharge q, and energy E computed by Schemes 1–3 using $\Delta x = 1/250$.

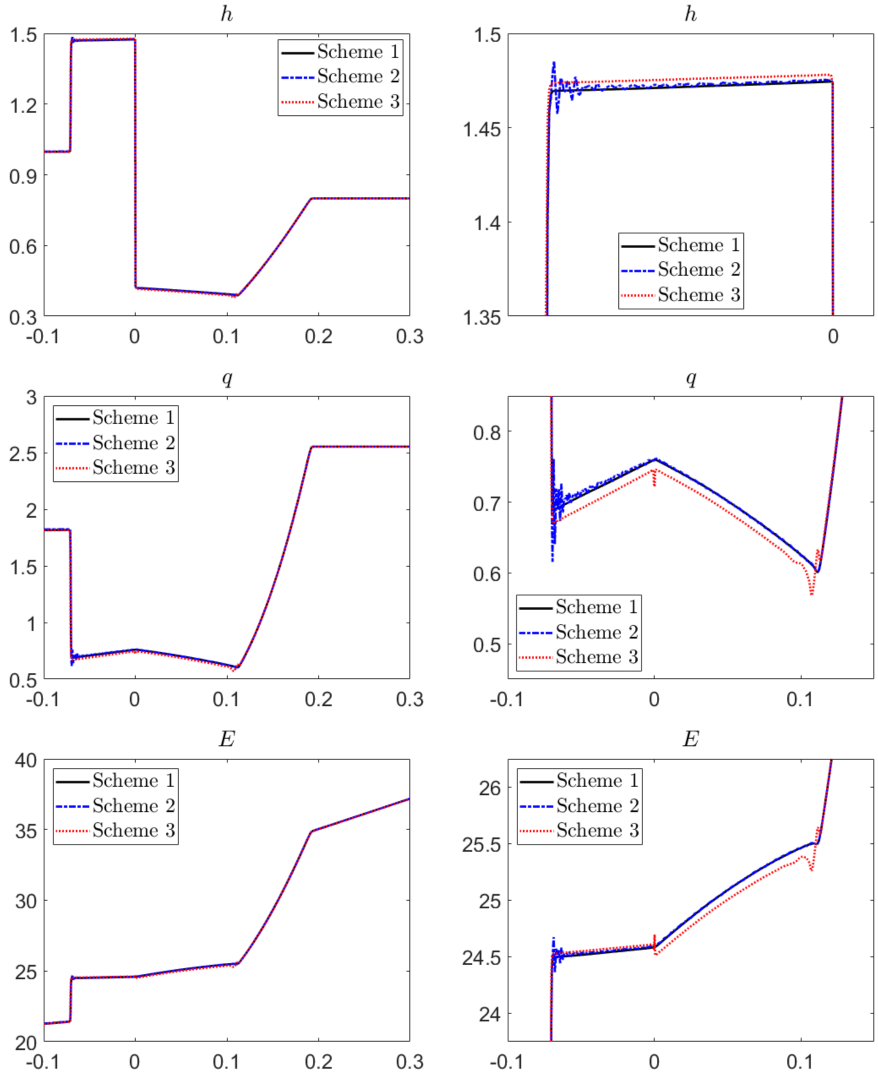

Figure 4.4: Example 3: Water depth h, discharge q, and energy E computed by Schemes 1–3 using $\Delta x = 1/2500$ (left column), and zoom at the areas containing oscillations in Scheme 2 solution (right column).

Example 4—Convergence to a Steady State $(n = 0.15)$

In this example, we study the convergence of the solutions computed by Schemes 1–3 towards the steady flow over a hump. We consider the continuous bottom topography given by

$$Z(x) = \begin{cases} 0.2 & \text{if } 8 \le x \le 12, \\ 0 & \text{otherwise,} \end{cases}$$

and the initial and boundary data that correspond to a subcritical flow:

$$h(x,0) \equiv 2 - Z(x), \quad q(x,0) \equiv 0, \quad q(0,t) = 4.42, \quad h(25,t) = 2,$$

with the boundary conditions for h at $x = 0$ and q at $x = 25$ set to be free.

We compute the numerical solutions until the final time $t = 500$ on the computational domain $[0, 25]$ covered by a uniform mesh with $\Delta x = 1/4$. The obtained numerical solutions ($h+Z$, q, and

E) are plotted in Figure 4.5. One can clearly see that the WB Schemes 1 and 2 converge to the constant q and E, whereas the non-WB Scheme 3 generates oscillations, whose size is proportional to the size of the local truncation error.

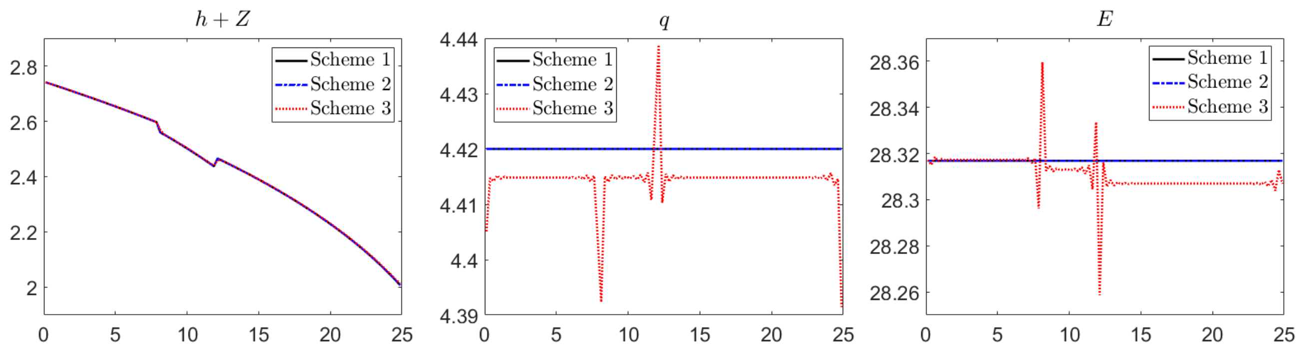

Figure 4.5: Example 4 (moving water steady state): $h+Z$, q, and E computed by Schemes 1–3.

We then test the ability of the studied schemes to capture the propagation of a small perturbation of the obtained moving-water equilibria. To this end, we denote the obtained steady states by $h_{\rm eq}(x)$ and $q_{\rm eq}(x)$ (notice that each scheme has its own discrete equilibrium), and then consider the following initial data:

$$h(x,0)=h_{\rm eq}(x)+\begin{cases}10^{-4}, & 9.5\le x\le 10.5,\\ 0, & \text{otherwise},\end{cases}\qquad q(x,0)=q_{\rm eq}(x).$$

We compute the solutions by the three studied schemes until the final time $t=1.5$ on the same uniform mesh with $\Delta x=1/4$. The obtained differences $h(x,1.5)-h_{\rm eq}(x)$ are plotted in Figure 4.6. One can clearly see that both the WB Schemes 1 and 2 can capture the time evolution of the perturbation quite accurately, whereas the non-WB Scheme 3 generates spurious oscillations as it can preserve still-water equilibria only. Even though Scheme 2 is WB, it still generates some oscillations, and thus Scheme 1 clearly outperforms both of its counterparts in this example.

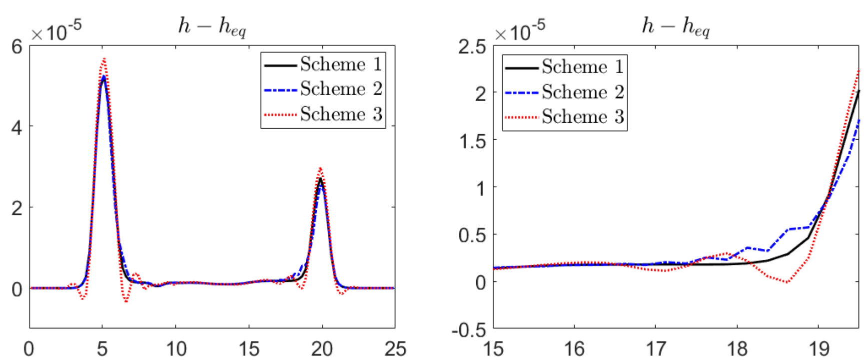

Figure 4.6: Example 4: The differences $h(x,1.5)-h_{\rm eq}(x)$ computed by Schemes 1–3 (left) and zoom at $x\in[15,19.5]$ (right).

4.3 Two-Layer Shallow Water System

Example 5—Small Perturbation of a Discontinuous Steady State

In the fifth example taken from [7, 14], we consider a discontinuous steady state given by

$$(h_1)_{\rm eq}(x) := \begin{cases} 1.22373355048230, & x<0,\\ 1.44970064153589, & x>0, \end{cases} \qquad (q_1)_{\rm eq}(x) \equiv 12,$$
$$(h_2)_{\rm eq}(x) := \begin{cases} 0.968329515483846, & x<0,\\ 1.12439026921484, & x>0, \end{cases} \qquad (q_2)_{\rm eq}(x) \equiv 10,$$

and a discontinuous bottom topography

$$Z(x) = \begin{cases} -2, & x<0,\\ -1, & x>0. \end{cases}$$

In order to test the ability of the studied schemes to capture quasi-steady solutions, we add a small perturbation to the upper layer depth and take the following initial data:

$$h_1(x,0) = (h_1)_{\rm eq}(x) + \begin{cases} 0.12, & x\in[-0.9,-0.8],\\ 0, & \text{otherwise}, \end{cases}$$
$$h_2(x,0) = (h_2)_{\rm eq}(x), \quad q_1(x,0) = (q_1)_{\rm eq}(x), \quad q_2(x,0) = (q_2)_{\rm eq}(x),$$

prescribed in the computational domain $[-1,1]$ subject to the free boundary conditions.

We compute the numerical solutions until the final time $t=0.08$ by Schemes 1–3 on a uniform mesh with $\Delta x = 1/100$. The differences $h_1(x,0.08)-(h_1)_{\rm eq}(x)$ and $h_2(x,0.08)-(h_2)_{\rm eq}(x)$ are plotted in Figure 4.7, where one can clearly see that, unlike the proposed Scheme 1, Schemes 2 and 3 produce oscillatory numerical results.

Example 6—Riemann Problem

In the last example taken from [7, 14], we numerically solve a Riemann problem with the following initial data:

$$(h_1,q_1,h_2,q_2)(x,0) = \begin{cases} (1,1.5,1,1), & x<0,\\ (0.8,1.2,1.2,1.8), & \text{otherwise}, \end{cases}$$

and discontinuous bottom topography:

$$Z(x) = \begin{cases} -2, & x<0,\\ -1.5, & \text{otherwise}, \end{cases}$$

prescribed in the computational domain $[-1,1]$ subject to free boundary conditions.

We compute the numerical solutions until the final time $t=0.1$ by Schemes 1–3 on a uniform mesh with $\Delta x = 1/50$. The obtained upper layer depth h_1 and lower layer depth h_2 are plotted in Figure 4.8, where one can see that Scheme 1 clearly outperforms Schemes 2 and 3 as there are no oscillations in the results computed by Scheme 1.

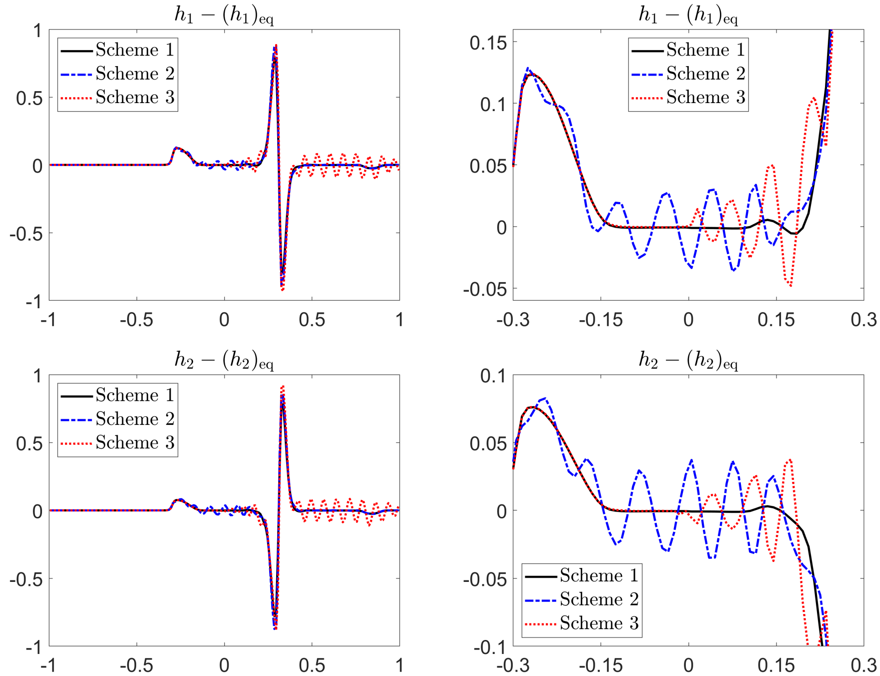

Figure 4.7: Example 5: The differences $h_1(x,0.08)-(h_1)_{\mathrm{eq}}(x)$ (top row) and $h_2(x,0.08)-(h_2)_{\mathrm{eq}}(x)$ (bottom row), and zoom at $x\in[-0.3,0.3]$ (right column).

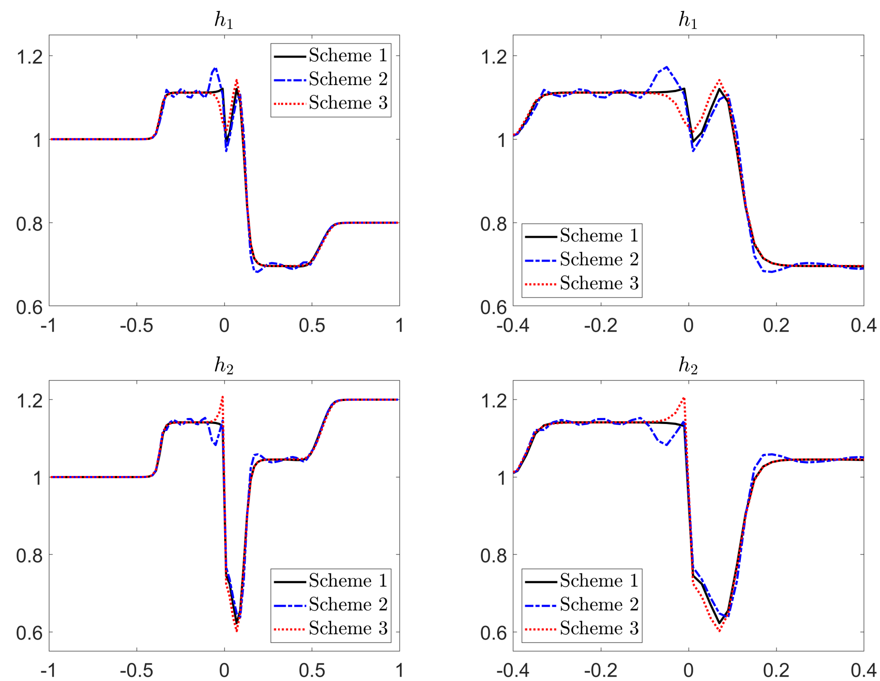

Figure 4.8: Example 6: Upper layer depth h_1 (top row) and lower layer depth h_2 (bottom row), and zoom at $x\in[-0.4,0.4]$ (right column).

4.4 1-D Euler Equations with Gravitation

Example 7—Shock Tube Problem

In the example taken from [18,27], we set $\phi(x)=x$ in the computational domain $[0,1]$, where we prescribe the following initial data:

$$(\rho(x,0),u(x,0),p(x,0))=\begin{cases}(1,0,1), & x\le 0.5,\\ (0.125,0,0.1), & x>0.5,\end{cases}$$

and the reflecting boundary conditions.

We compute the numerical solutions until the final time $t=0.2$ by Schemes 1 and 2 on a uniform grid with $\Delta x=1/50$. The numerical results are shown in Figure 4.9, where one can clearly see that Scheme 1 outperforms Scheme 2 as there are no oscillations in the results computed by Scheme 1, and these results are in good agreement with those reported in [18, 27].

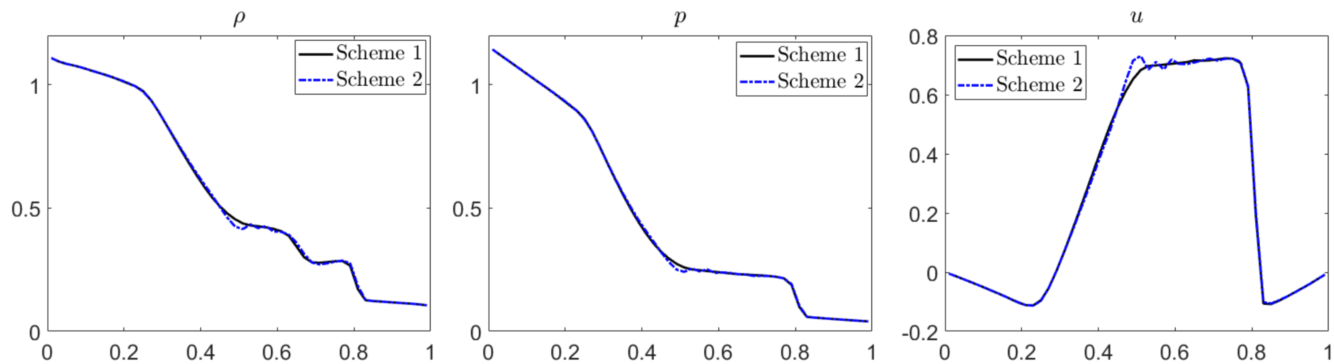

Figure 4.9: Example 7: ρ, p, and u computed by Schemes 1 and 2.

4.5 2-D Euler Equations with Gravitation

Example 8—Discontinuous Perturbation of a Steady State

In this 2-D example, which is a modification of the example studied in [4, 27], we consider the following hydrostatic equilibrium:

$$\rho_{\rm eq}(x,y)=1.21e^{-1.21\phi(x,y)},\quad u_{\rm eq}(x,y)=v_{\rm eq}(x,y)\equiv 0,\quad p_{\rm eq}(x,y)=e^{-1.21\phi(x,y)},$$

with the gravitational potential $\phi(x,y)=x+y$. We take the computational domain $[0,1]\times[0,1]$ and impose the free boundary conditions.

We construct the discrete steady state as it was described in [15, Example 7], but with the integrals in K^x and K^y evaluated within the fifth order of accuracy.

Equipped with the discrete steady state, we first numerically verify that Schemes 1 and 2 can preserve it within the machine accuracy, and then introduce a discontinuous pressure perturbation and consider the perturbed initial data:

$$\rho(x,y,0)=\rho_{\rm eq}(x,y),\quad u(x,y,0)=v(x,y,0)\equiv 0,\quad p(x,y,0)=p_{\rm eq}(x,y)+\begin{cases}0.5, & x^2+y^2\le 0.15^2,\\ 0, & \text{otherwise}.\end{cases}$$

We compute the numerical solutions by Schemes 1 and 2 on a uniform grid with $\Delta x=\Delta y=1/80$ at time $t=0.12$ and plot the obtained results in Figure 4.10. As one can see, Scheme 1 clearly outperforms Scheme 2 as there are almost no oscillations in the results computed by Scheme 1.

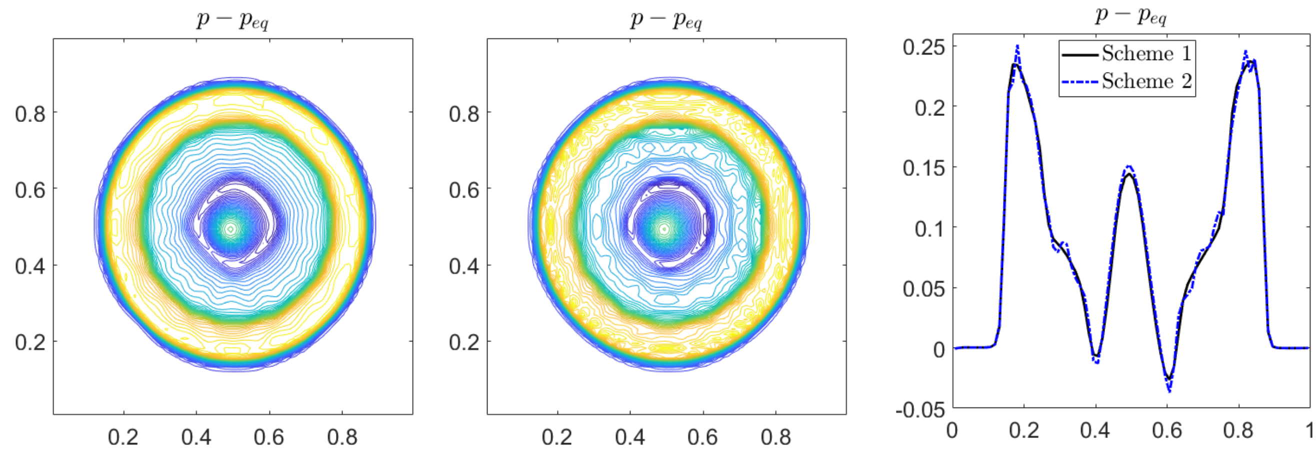

Figure 4.10: Example 8: Pressure perturbation $(p(x,y,0.12) - p_{\rm eq}(x,y))$ captured by Schemes 1 (left) and 2 (middle) and their 1-D slices along $y = 0.5$ (right).

Remark 4.1 *In the above examples, we have shown that reconstructing equilibrium variables through the LCD is advantageous as it helps to remove WENO-type oscillations while keeping the scheme WB. In the 1-D case, the proposed LCD of the equilibrium variables is based on the projections onto the eigenvectors of the matrices $C_j = C(\boldsymbol{U}_j)$ as explained in §3. Instead, one may try to base the LCD of equilibrium variables on the eigenvectors of the matrices $\mathcal{A}_j = \mathcal{A}(\boldsymbol{U}_j) := \partial \boldsymbol{F}/\partial \boldsymbol{U}(\boldsymbol{U}_j) - B(\boldsymbol{U}_j)$. To this end, one needs to compute the matrices $\widehat{Q}_j$ and $\widehat{Q}_j^{-1}$ such that $\widehat{Q}_j^{-1}\mathcal{A}_j\widehat{Q}_j$ is a diagonal matrix, and then to apply the Ai-WENO-Z interpolation to a different set of the local characteristic variables $\widehat{\boldsymbol{\Gamma}}_\ell$, which are defined by*

$$\widehat{\boldsymbol{\Gamma}}_\ell = \widehat{Q}_j^{-1}\boldsymbol{E}_\ell, \quad \ell = j \pm 2, j \pm 1, j.$$

The Ai-WENO-Z reconstruction then gives $\widehat{\boldsymbol{\Gamma}}^\pm_{j\mp\frac12}$ and hence $\boldsymbol{E}^\pm_{j\mp\frac12} = \widehat{Q}_j\boldsymbol{\Gamma}^\pm_{j\mp\frac12}$, which are different from the values $\boldsymbol{E}^\pm_{j\mp\frac12}$ obtained in (3.4)–(3.5).

However, this kind of LCD does not lead to very good results. To demonstrate this, we recompute the numerical solutions in Examples 2, 4, and 5 using Scheme 1, but with an aforementioned alternative LCD based on $\mathcal{A}(\boldsymbol{U})$ rather than $C(\boldsymbol{U})$. The obtained results shown in Figures 4.11–4.14 confirm the advantages of the LCD based on $C(\boldsymbol{U})$.

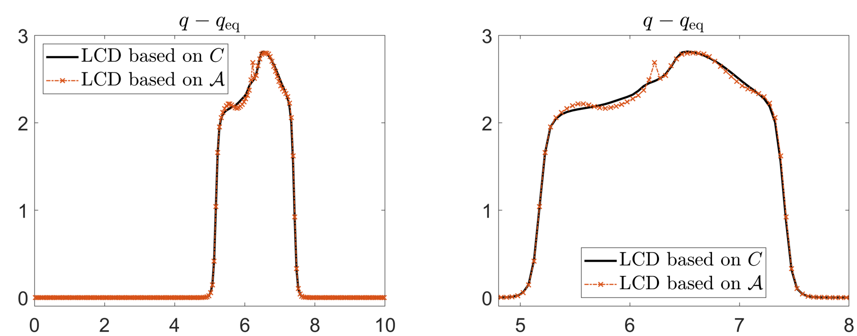

Figure 4.11: Example 2: The differences $q(x,0.5) - q_{\rm eq}(x)$ computed by Scheme 1 with two different LCD of the equilibrium variables (left) and zoom at $x \in [4.8, 8]$ (right).

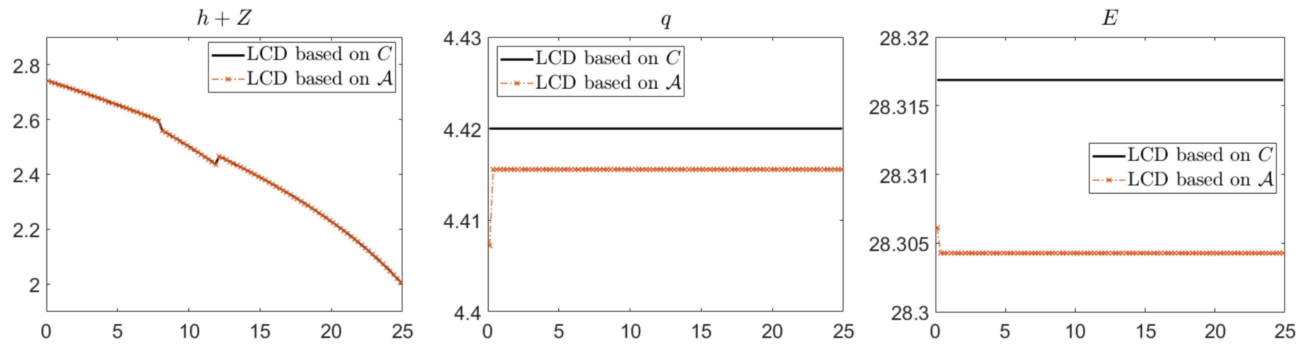

Figure 4.12: Example 4 (moving water steady state): $h+Z$, q, and E computed by Scheme 1 with two different LCD of the equilibrium variables.

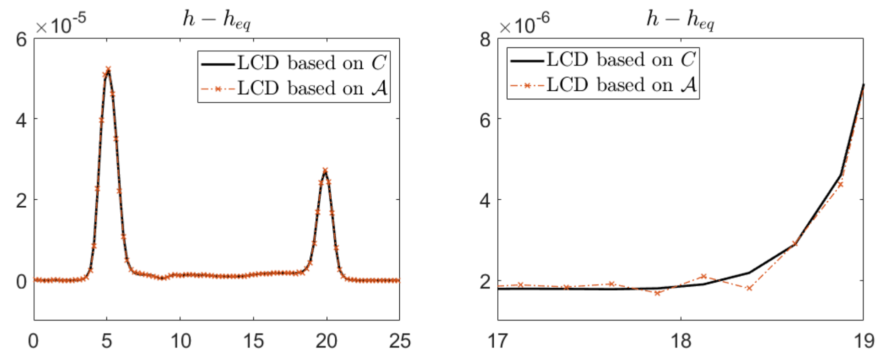

Figure 4.13: Example 4: The differences $h(x,1.5)-h_{\rm eq}(x)$ computed by Scheme 1 with two different LCD of the equilibrium variables (left) and zoom at $x\in[17,19]$ (right).

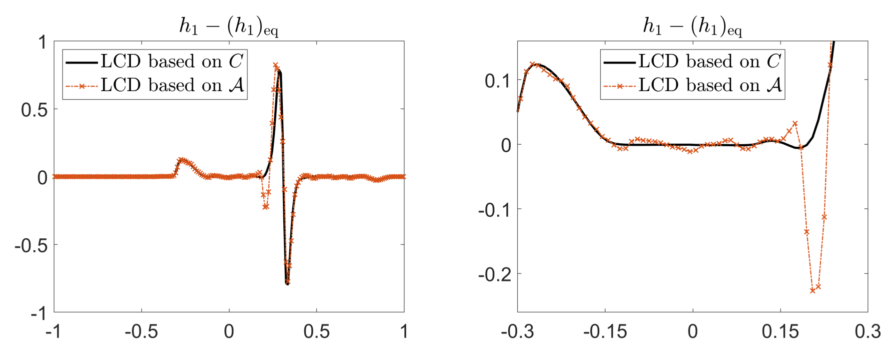

Figure 4.14: Example 5: The differences $h_1(x,0.12)-(h_1)_{\rm eq}$ computed by Scheme 1 with two different LCD of the equilibrium variables (left) and zoom at $x\in[-0.3,0.3]$ (right).

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